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THE ELEMENTARY TREATMENT OF CONICS BY MEANS OF THE REGULUS.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

(Read before the American Mathematical Society, February 25, 1905.)

INASMUCH as projective plane geometry depends on properties of space (since the theorem on triangles in perspective in a plane cannot be established by plane geometry without the use of metric concepts), there is no logical objection to the employment of space considerations in proving theorems of plane geometry. The regulus supplies extremely simple proofs of the properties of a conic, whether this be considered as a locus or an envelope; these proofs have the advantage of connecting the points of a conic and the tangents of a conic from the first, instead of leaving the connection to be proved later by an elaborate chain of reasoning. The method makes unnecessary also the treatment of the tangent as the limiting position of a chord when the determining points become indistinguishable, which is open to some objections in pure geometry; not on account of the assumption as to continuity, for this has already been admitted in the proof of the fundamental theorem of projective geometry, but with regard to elegance and directness of proof. This note contains the application of the method to the proofs of the theorems of Chasles, Brianchon, and Pascal, and of polar and involution properties.

The regulus is the system of lines (rays) that meet three non-incident lines, the directors. By means of the fundamental theorem of projective geometry it is proved that the regulus is crossed by a second regulus; any three rays of either serve as directors of the other; through any point on a ray of either there passes a ray of the other.

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Three directors, u, v, w, and three rays a, b, c (that is, two incident triads of lines) give rise to an interesting figure on which the proofs depend. The pairs of lines au, bv, cw determine points A, B, C and planes α, β, γ; the three planes meet in a point O, the three points lie in a plane ω. The point $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$ and the plane $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$, that is, O and ω, are pole and polar. The pairs of points (bw, cv), (cu, aw), (av, bu), that is, $\begin{smallmatrix} b & & c \\ & \times & \\ v & & w \end{smallmatrix}$, $\begin{smallmatrix} c & & a \\ & \times & \\ w & & u \end{smallmatrix}$, $\begin{smallmatrix} a & & b \\ & \times & \\ u & & v \end{smallmatrix}$, lie on lines through O, namely, the lines $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$; the pairs of planes $\begin{smallmatrix} b & & c \\ & \times & \\ v & & w \end{smallmatrix}$, $\begin{smallmatrix} c & & a \\ & \times & \\ w & & u \end{smallmatrix}$, $\begin{smallmatrix} a & & b \\ & \times & \\ u & & v \end{smallmatrix}$ meet in lines in ω, namely, BC, CA, AB. The points are named as shown in Fig. 1, the planes are named to correspond, *e. g.*, the point av is A', the plane av is α'; the plane α', β', γ' determine th point O', etc.

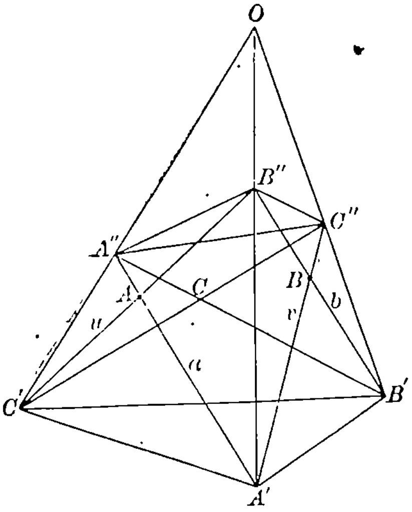

FIG. 1.

The triangles $A'B'C'$, $B''C''A''$ are perspectiv from O; $A''B''C''$, BCA from O'; ABC, $B'C'A'$ from O''. It is easily shown that O, O', O'' are collinear, ω, ω', ω'' are coaxial; and that if the rays and directors are associated in the reversed order $\begin{pmatrix} a & c & b \\ u & v & w \end{pmatrix}$, the three new centres of perspective lie on the axis of $\omega\ \omega'\ \omega''$, and the axis of the new planes is $O\ O'\ O''$. These facts, however, are not needed for the present purpose.

It is at once seen that O, ω are harmonic to the pairs of lines au, bv, cw, that is, to the two triads of rays a, b, c and directors u, v, w, hence to the complete system of transversals to a, b, c (directors) and transversals to u, v, w (rays). This shows that the rays and directors intersect in pairs at points on ω, lie in pairs (the same pairs) on planes through O; the two of a pair d, x thus associated are harmonic with regard to O and ω. Any chord through O joins a point on a ray d to a point on the associated director x, and is therefore divided harmonically by O, ω. Moreover, if O be any point on a chord that meets a ray d and director x, then the point dx lies on the polar plane of O.

Any line p, not a director, that meets one ray meets one other, but no more. For let p pass through A (*i. e.*, au); the plane pu meets v, w, at V, W; the line VW is therefore a ray, and is met by p. The only possible exception arises when VW is itself the ray a; then the line p meets the regulus at the point A only (or, the line p meets only one ray of the regulus); p is a tangent line, and the plane α (*i. e.*, au), in which lie all tangent lines through the point A (*i. e.*, au), is a tangent plane. The facts proved above as to the relation of O, ω may be stated in the form: The points of contact of tangent planes from the point $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$ lie on the plane $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$.

In order that a tangent plane may contain a line q, it must contain any two points S, S' on the line; hence the point of contact must lie on both σ and σ', that is on a line q'; q, q' are conjugate lines. Hence if there is one tangent plane through q, there is precisely one other, unless q' is itself a ray or director, which happens only when q is a ray or director, and then q' coincides with q.

The ranges of points determined on two directors u, v by the rays $a, b, c, d, \cdots$ are sections of the axial pencil $x.abcd \cdots$ (where x is any other director) by the transversals u, v; hence they are projective. The regulus is therefore the system of lines that join corresponding points of projective ranges on two non-incident lines. Again, the axial pencils $u.abcd \cdots$, $v.abcd \cdots$ have a common section by the transversal x, hence they are projective. The regulus is therefore the system of lines determined by corresponding planes of two projective axial pencils whose axes are non-incident.

Let the regulus be cut by a plane ω; each ray is thus associated with a particular director; the two, d, x, meet in a point D on ω, and lie in a plane δ (a tangent plane) through O. The projective axial pencils $u.abcd \cdots$, $v.abcd \cdots$ are cut in projective flat pencils, centers A and B; hence the plane section of a regulus (which is a point system of the second order, since a line through A cuts one other ray) is the locus of the intersection of corresponding rays of two projective flat pencils. Consider also the line system composed of the tangent lines in the plane ω; these are the intersections of the tangent planes au, bv, cw, dx, ey, etc. by ω, they are therefore the projections of rays $a, b, c, d, e, \cdots$ (or of directors $u, v, w, x, y, \cdots$) from O on ω; call them $a', b', c', \cdots$. Since the ranges $u.abcd\cdots$, $v.abcd\cdots$ are projective, their projections from O on ω are projective. Hence the lines $a', b', c', d', \cdots$ in the plane ω connect the corresponding points of projective ranges on the two tangents a', b'; that is, the line system (which, by what has been said about tangent planes, is seen to be of the second order) is composed of the lines that join corresponding points of projective ranges. Moreover, as to the relation of the two systems: the flat pencil $A.ABCD\cdots$ in the plane $\omega =$ the axial pencil $u.abcd\cdots =$ range $v.abcd\cdots =$ range $v'.a'b'c'd'\cdots$ in the plane ω. Thus the section (aggregate of points and lines) has the property that the pencil subtended at any point A by the points C, D, E, F is projective with the range determined on any tangent b' by the tangents c', d', e', f', which is Chasles's theorem. The section will now be called a conic.

Since the pencil determined by four points is projectively the same wherever on the section its vertex may lie, and similarly for the range determined by four tangents, it is unnecessary to specify the vertex or base; it is sufficient to speak of the four points or four lines. Since also the range determined by four rays on any director of the regulus depends only on the rays, it is sufficient to speak of the four rays. What has been proved above can be stated in the form: Four points of a conic are projective with the four rays, or four directors, that pass through them; four tangents to a conic are projective with the four rays, or four directors, that pass through their points of contact.

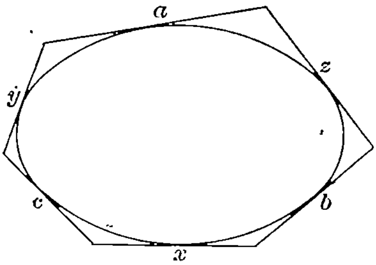

FIG. 2.

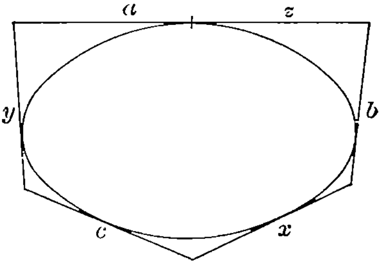

FIG. 3.

If four rays a, b, c, d are projective with four directors u, v, w, x, their intersections are coplanar. For let y be the director through the point D in which the plane au, bv, cw (that is, ABC) meets d; then the rays a, b, c, d are projective with the points A, B, C, D, and these are projective with the directors u, v, w, y. Hence $u\,v\,w\,x = u\,v\,w\,y$, and y is therefore the same as x.

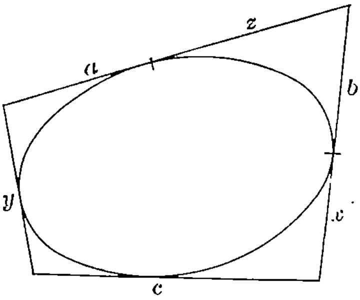

FIG. 4.

The polar properties follow at once from the harmonic relation borne to the rays and directors by O, ω. A point T in the plane ω has a polar plane τ, which by harmonic symmetry of the whole figure to $O\omega$ must pass through O. The section of τ by ω is the polar line of T with regard to the section considered, and it has already been shown that any chord through T, and therefore a chord of the section, is harmonically divided by $T\tau$. Hence follows the usual quadrilateral construction for pole and polar.

To prove Brianchon's theorem, take the six tangents as projections alternately of rays and directors, as shown in Fig. 2. In space, the lines $\overset{b\quad c}{\underset{y\quad z}{\times}}$, $\overset{c\quad a}{\underset{z\quad x}{\times}}$, $\overset{a\quad b}{\underset{x\quad y}{\times}}$ meet at a point S; projecting the figure from O on to ω, we obtain three concurrent lines joining opposite vertices of the circumscribing hexagon.

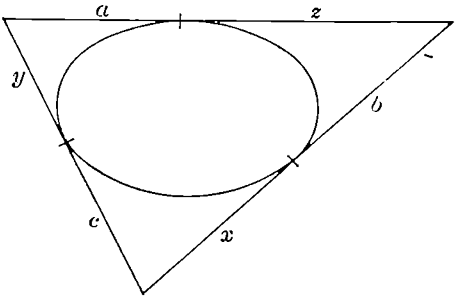

FIG. 5.

To prove the special cases of this theorem for the circumscribing pentagon, quadrilateral, triangle, regard the necessary number of tangents (one, two or three) as projections of both a ray and a director, as shown in Figs. 3, 4, 5; the proof then applies without change, the point az being the point of contact of the tangent.

To prove Pascal's theorem, take alternately the rays and directors that pass through the six points, Fig. 6. The common lines of the planes $\overset{b\ c}{\underset{y\ z}{\times}}$, $\overset{c\ a}{\underset{z\ x}{\times}}$, $\overset{a\ b}{\underset{x\ y}{\times}}$ lie in a plane σ; hence in the section by the plane ω the common points of the lines $\overset{b\ c}{\underset{y\ z}{\times}}$, $\overset{c\ a}{\underset{z\ x}{\times}}$, $\overset{a\ b}{\underset{x\ y}{\times}}$ lie in a line, the line $\sigma\omega$. The special cases of this theorem, which arise when sides of the inscribed hexagon (one, two, or three) are replaced by tangents, require no change in the proof. Moreover, the same proof applies when the section by the plane ω consists of a ray and a director, that is, when the theorem to be proved becomes Pappus's theorem, that if the vertices of a hexagon are taken to lie alternately on two intersecting lines, the three intersections of opposite sides are collinear.

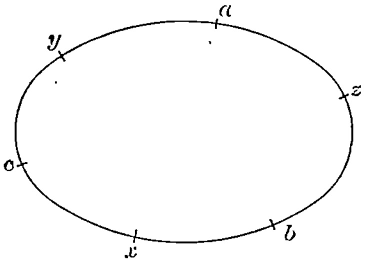

FIG. 6.

It has been mentioned that if a chord meets d, x, the point dx lies on the polar plane of any point on the chord. Hence if chords of a conic, A_1A_2, B_1B_2, C_1C_2, $\cdots$, are concurrent in S, then $A_1B_1C_1D_1 = A_2B_2C_2D_2$; for let a, b, c, d be the rays through A_1, B_1, C_1, D_1, and u, v, w, x the directors through A_2, B_2, C_2, D_2, and let the points au, bv, cw, dx (which are known to lie on σ) be U, V, W, X. Then $A_1B_1C_1D_1 = abcd = UVWX$ (in the section σ) $= uvwx = A_2B_2C_2D_2$. This proof is unaffected by coincidence; D_1D_2 may be the same as A_2A_1. Thus $A_1B_1C_1A_2 = A_2B_2C_2A_1$. Conversely, if in the plane ω $A_1B_1C_1A_2 = A_2B_2C_2A_1$, the chords A_1A_2, B_1B_2, C_1C_2 are concurrent. For since $A_1B_1C_1A_2 = abcd$, and $A_2B_2C_2A_1 = uvwx$, we have $abcd = uvwx$, and therefore by a result obtained earlier, the points au, bv, cw, dx lie on a plane σ. Hence the planes au, bv, cw, dx meet in a point S. But the planes au, dx meet in a line in the plane ω, hence the point S lies in ω. Since then the planes

au, bv, cw, dx meet at a point S in ω, the section of these planes by ω gives chords of the conic concurrent in S. Thus we have the theorem : If points of a conic are projectively paired, they lie on concurrent chords, which is the foundation of the theory of involution on a conic.

For the sake of brevity, as little detail as possible has been given in this note, the design being simply to draw attention to this mode of proving fundamental properties of the conic.

BRYN MAWR COLLEGE,
June, 1905.

ARZELÀ'S CONDITION FOR THE CONTINUITY OF A FUNCTION DEFINED BY A SERIES OF CONTINUOUS FUNCTIONS.

BY PROFESSOR E. J. TOWNSEND.

(Read before the American Mathematical Society, September 7, 1905.)

§ 1. A FUNCTION defined by a series whose terms are continuous functions may or may not be itself continuous. It may in fact be discontinuous at a set of points everywhere dense within the interval of definition. It is important to establish a criterion by which the continuity of such a function may be determined. Conditions which are sufficient, although not necessary, are to be found in any extensive work on calculus. Arzelà was the first to formulate a set of conditions which are both necessary and sufficient.* In his first discussion of the subject, however, he was not sufficiently rigorous.† A later and more rigorous development was given, differing from the first in some particulars.‡ Still more recently he has revised his first set of proofs and maintains that they are now sufficiently rigorous to be valid.§ It is the purpose of this paper to present in substance the final results of Arzelà's investigations.

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*Intorno alla continuità della somma di infinite funzioni continue, Bologna, 1884.

†See Schoenflies, Punktmannigfaltigkeiten, p. 225, footnote. Also Arzelà, Sulla serie di funzioni di variabili reali, Bologna, 1902, p. 6.

‡Sulla serie di funzioni, part 1, Bologna, 1899, p. 10 et seq.

§See Sulla serie di funzioni di variabili reali, Bologna, 1902.

Let us then define $S_n(x)$ by the relation

$$S_n(x) = u_1(x) + u_2(x) + \cdots + u_\kappa(x) + \cdots + u_n(x), \quad a \leqq x \leqq \beta,$$

where $u_\kappa(x)$ and hence $S_n(x)$ is a continuous function of x. Moreover, let us define $f(x)$ by the equation

$$f(x) = \underset{n=\infty}{\mathrm{L}} S_n(x), \qquad a \leqq x \leqq \beta.$$

The interval of convergence of the series is then the interval of definition of the function $f(x)$. The problem is to find a set of conditions which are necessary and sufficient for the continuity of $f(x)$ in a given interval (a, β), equal to or less than the interval of convergence.

With Arzelà, let us consider this problem as a special case of the following more general problem. Suppose $f(x, y)$ to be a function of the two independent variables x and y, defined for all values of x within the closed interval (a, β) and for

$$y \equiv (y) = y_1, y_2, y_3, \cdots, y_s, \cdots,$$

$y_1, y_2, y_3, \cdots, y_s, \cdots$ being a set of values dense at $y = y_0$, but not including y_0. Let $f(x, y_s)$ be a continuous function of x for each value y_s included in the set (y). For $y = y_0$, let $f(x, y)$ be defined by the equation

$$f(x, y_0) = \underset{y_s=y_0}{\mathrm{L}} f(x, y_s).$$

The more general problem, which we shall now consider, is to find a necessary and sufficient set of conditions that the limiting function $f(x, y_0)$ shall be a continuous function of x in the interval $a \leqq x \leqq \beta$. If we put $y_s = 1/n$, $n = 1, 2, 3, \cdots$, this reduces at once to the case of the infinite series given above.*

§ 2. First of all, let us consider the necessary and sufficient condition that the limiting function $f(x, y_0)$ shall be continuous at a single point x_0 of the interval (α, β). This condition may be stated as follows:

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*The results given in this paper can be obtained, and perhaps more easily, without the introduction of the more general problem indicated. It is sufficient for this purpose to consider $S_n(x)$ as a function of the two variables x, n and to make of use the principles of ordinary convergence. Arzelà's method has been retained, not because it has any virtue in itself, but because it gives an opportunity to interpret more clearly his results at a point where he has been criticized.

I. *In order that the function* $f(x, y_0)$, *defined as in* § 1 *for the interval* (α, β), *be continuous at any point* x_0 *in this interval, it is necessary and sufficient that to an arbitrarily small positive number* σ *and to every* y_t *sufficiently near* y_0 *there shall correspond a neighborhood of the point* x_0, *which, however, may vary in extent with* y_t, *such that for every value of* x *in this neighborhood*

$$|f(x, y_0) - f(x, y_t)| < \sigma.$$

We shall first show that the above condition is necessary. Let us suppose then that $f(x, y_0)$ is a continuous function of x at the point $x = x_0$. By virtue of the limit

$$f(x_0, y_0) = \underset{y_s \doteq y_0}{\mathrm{L}} f(x_0, y_s),$$

there must exist a definite number y_{s_1} of the set (y) such that for every y_t of this set lying between y_{s_1} and y_0, we have

$$(1) \qquad |f(x_0, y_0) - f(x_0, y_t)| < \sigma/3,$$

where σ as elsewhere in this paper is an arbitrarily small positive number.

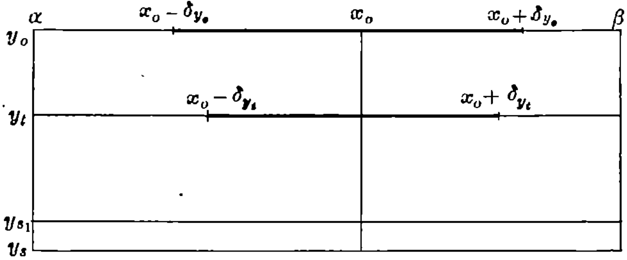

FIG. 1.

Because $f(x, y_0)$ is a continuous function of x for $y = y_0$, we have also for some neighborhood of x_0, say $(x_0 - \delta'_{y_0}, x_0 + \delta_{y_0})$, the inequality

$$(2) \qquad |f(x, y_0) - f(x_0, y_0)| < \sigma/3.$$

Likewise, because $f(x, y_t)$ is a continuous function of x, there exists a neighborhood $(x_0 - \delta'_{y_t}, x_0 + \delta_{y_t})$ on the line $y = y_t$ (Fig. 1), such that for every x within it

$$(3) \qquad |f(x_0, y_t) - f(x, y_t)| < \sigma/3.$$

Combining (1), (2), (3), we have

$$|f(x, y_0) - f(x, y_t)| < \sigma,$$

which holds for every y_t between y_{s_1} and y_0 and for every x within a neighborhood $(x_0 - d'_{y_t}, x_0 + d_{y_t})$, where d'_{y_t} is the smaller of the two values δ'_{y_0} and δ'_{y_t}, and d_{y_t} is the smaller of the two values δ_{y_0} and δ_{y_t}.

The given condition may be shown to be sufficient as follows. We have by hypothesis the inequality

$$(4) \qquad |f(x, y_t) - f(x, y_0)| < \sigma,$$

which is valid for every value of x within a certain neighborhood, say $(x_0 - \delta', x_0 + \delta)$. The existence of the limit

$$f(x_0, y_0) = \mathop{\mathrm{L}}_{y_t \doteq y_0} f(x_0, y_t)$$

gives, for all values of y_t from a certain definite value on, the inequality

$$(5) \qquad |f(x_0, y_0) - f(x_0, y_t)| < \sigma.$$

By hypothesis, $f(x, y_t)$ is a continuous function of x for each value of y_t. Hence, for every x within a certain neighborhood, say $(x_0 - \delta'_1, x_0 + \delta_1)$, we have

$$(6) \qquad |f(x_0, y_t) - f(x, y_t)| < \sigma.$$

By combining the inequalities (4), (5), and (6), we have

$$|f(x_0, y_0) - f(x, y_0)| < 3\sigma,$$

which holds for all values of x lying within the smaller of the two neighborhoods $(x_0 - \delta', x_0 + \delta)$ and $(x_0 - \delta'_1, x_0 + \delta_1)$. This inequality establishes the continuity of $f(x, y_0)$ at the point $x = x_0$ as the theorem requires.

§ 3. Let us now consider the necessary and sufficient condition that $f(x, y_0)$ shall be a continuous function of x throughout the closed interval (α, β). This condition Arzelà states in substance as follows:

II. *In order that the function $f(x, y_0)$ defined as in § 1 be continuous throughout the interval $\alpha \leqq x \leqq \beta$, it is necessary and* *sufficient that to an arbitrarily small positive number* σ *and to an arbitrary* $y_s \neq y_0$, *there shall correspond a positive number* l *and a finite set of values*

$$y = y_{s_1}, y_{s_2}, \cdots, y_{s_p}$$

lying between y_s *and* y_0, *for which the following condition holds: On lines* $y = y_{s_t} (t = 1, 2, \cdots, p)$, *it shall be possible to choose a series of segments, each of length at least equal to* l, *whose projections on the line* $y = y_0$, *when taken together, completely cover the interval* $\alpha \leqq x \leqq \beta$, *and for which, furthermore, the relation*

$$|f(x, y_0) - f(x, y_{s_t})| < \sigma$$

holds, (x, y_{s_t}) *being an arbitrary point on any one of these segments.*[*]

It may be shown as follows that this condition is necessary.[†] Let us assume then that $f(x, y_0)$ is a continuous function of x throughout the interval $\alpha \leqq x \leqq \beta$. By virtue of theorem I, there exists for any $x' (\alpha \leqq x' \leqq \beta)$ and for each y_t sufficiently near y_0 a neighborhood $(x' - \delta'_{y_t}, x' + \delta_{y_t})$, which may vary with y_t but always, however, in such a manner that for each value of x within it the inequality

$$(1) \qquad |f(x, y_0) - f(x, y_t)| < \sigma$$

is valid for any previously assigned positive value of σ. In this discussion Arzelà distinguishes between that part of the neighborhood to the right of x' and that part lying to the left. Let us denote these two parts of the neighborhood by $\Delta(x', y_t)$ and $\Delta'(x', y_t)$ respectively. These two parts may or may not be of the same magnitude. In other words, having chosen an arbitrarily small positive number σ and a y_s at pleasure, we may then select any value x' of the interval (α, β), end points included, and there will exist upon some line $y = y_t$ lying between $y = y_s$ and $y = y_0$ a neighborhood of magnitude $\Delta(x', y_t)$ to the right of x', such that for every x within it the inequality (1) is valid. Let us now consider $\Delta(x', y_t)$ for all values of y_t included in the set (y) and lying between y_0 and y_s. For some of these values of y_t, $\Delta(x', y_t)$ may be zero, and consequently for these particular values of y_t no neighborhood exists to the right of x' for which the above inequality holds. It may happen that for some values of y_t lying between y_0 and y_s this inequality does not hold even for x' itself. However $\Delta(x', y_t)$ cannot be zero for all values of y_t lying between the above limits unless x' is the extreme point β; for we know that when y_t is taken sufficiently near y_0 $\Delta(x', y_t)$ is always greater than zero. Having selected x', there may exist for any particular y_t a great variety of values which might be taken as the corresponding $\Delta(x', y_t)$. In what follows, let us understand by $\Delta(x', y_t)$ the upper limit of all these values. This magnitude $\Delta(x', y_t)$, regarded as a function of y_t for all values of y_t between y_s and y_0, has an upper limit greater than zero, providing x' is different from β. Such a superior limit is therefore definitely determined for each value of x', σ and y_s having been previously selected. Let us denote this upper limit by $\Delta(x')$. In the same way, consider that part of the interval $(x' - \delta'_{y_t}, x' + \delta_{y_t})$ lying to the left of x', and denote by $\Delta'(x')$ the corresponding upper limit, which must likewise be different from zero for x' different from α.

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[*] In his statement of the theorem, Arzelà does not say that the given interval shall be closed. This, however, is necessary.

[†] See Sulle serie di funzioni, Bologna, 1899, p. 17, et seq.

Consider now the sum

$$\Delta'(x') + \Delta(x').$$

This sum is uniquely determined for each value x' of x, where $\alpha \leqq x \leqq \beta$. As a function of x, it has a lower limit which we shall now show to be greater than zero. Denote this lower limit by l. There must exist then in (α, β) at least one point, in every neighborhood of which the lower limit of $\Delta'(x) + \Delta(x)$ is l. Let x_1 be such a point. For the point x_1 itself, we have the value $\Delta'(x_1) + \Delta(x_1)$, which is certainly greater than zero. Among the values of (y) included between y_s and y_0, there is at least one, say y_t, for which the magnitude $\Delta(x_1, y_t)$ of the neighborhood to the right of x_1, within which for each value of x we have

$$|f(x, y_0) - f(x, y_t)| < \sigma,$$

is as near $\Delta(x_1)$ as we may choose. In the same way, there exists a y_r, which may be equal to or different from y_t, such that the magnitude $\Delta'(x_1, y_r)$ of the neighborhood to the left of x_1, within which we have for each value of x

$$|f(x, y_0) - f(x, y_r)| < \sigma,$$

is as near $\Delta'(x_1)$ as we choose to make it.

On the line $y = y_0$, consider the sub-interval

$$\left(x_1 - \frac{\Delta'(x_1, y_r)}{2}, x_1 + \frac{\Delta(x_1, y_t)}{2}\right).$$

For every value x'' included within this interval, there exists the neighborhood (Fig. 2)

$$\left(x_1, x_1 + \frac{\Delta(x_1, y_t)}{2}\right)$$

or the neighborhood

$$\left(x_1 - \frac{\Delta'(x_1, y_r)}{2}, x_1\right),$$

according as x'' is to the right or to the left of x_1, such that for any value of x within these neighborhoods we have either

$$|f(x, y_0) - f(x, y_t)| < \sigma$$

or

$$|f(x, y_0) - f(x, y_r)| < \sigma.$$

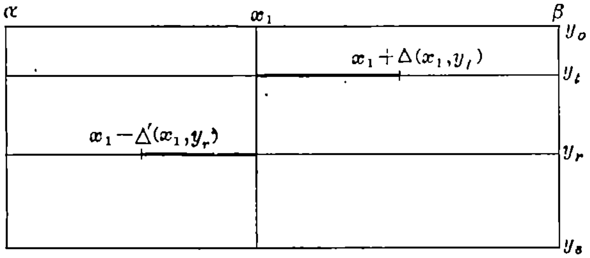

FIG. 2.

Consequently, for every value x'' the corresponding value of $\Delta'(x) + \Delta(x)$ is greater than the smaller of the two numbers

$$\frac{\Delta'(x_1, y_r)}{2}, \quad \frac{\Delta(x_1, y_t)}{2}.$$

The lower limit of $\Delta'(x) + \Delta(x)$, namely l, must then be greater than or at least equal to the smaller of the same two numbers. But we have already seen that these numbers can be made to differ as little as we choose from

$$\frac{\Delta'(x_1)}{2}, \qquad \frac{\Delta(x_1)}{2},$$

respectively, and these latter are certainly greater than zero. Hence we have in any case

$$l > 0.$$

Since the lower limit of $\Delta'(x) + \Delta(x)$ for $\alpha \leqq x \leqq \beta$ is greater than zero, it can be shown that a finite number of segments of the lines lying between $y = y_0$ and $y = y_1$ can be found fulfilling the requirements of the theorem. The projections of these segments upon $y = y_0$ may, and in general will, overlap. That a finite number of them is sufficient in order that the sum of their projections shall completely fill the interval (α, β) may be shown by considering the end points of those parts which the projections contribute to the sum. If there are an infinite number of such points, then there must exist on the line $y = y_0$ at least one limiting point of the corresponding values of x. This, however, can be avoided by the proper selection of the segments. For, let x_0 be such a limiting point. As we have seen, the value of $\Delta'(x) + \Delta(x)$ for $x = x_0$ is at least equal to l. Hence by the proper selection of a segment, we have for x_0 a neighborhood on $y = y_0$ equal to or greater than l which is entirely free from the end points mentioned above, and this is contrary to the supposition that they must of necessity become dense at some point.

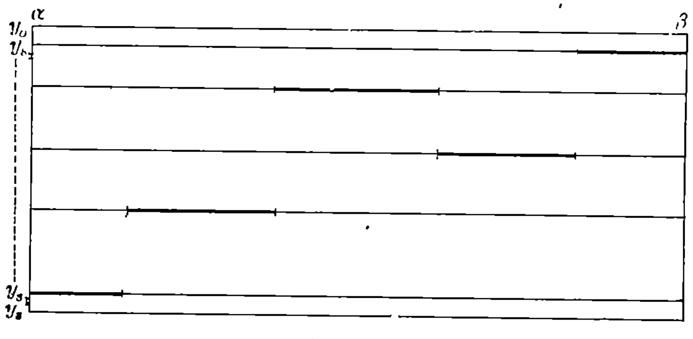

FIG. 3.

It follows then that having selected an arbitrarily small positive number σ and a y_s at pleasure, we may choose a number $l > 0$ and lay off on the lines

$$y = y_{s_1},\ y_{s_2},\ \cdots\ y_{s_p},$$

$(y_{s_1}, y_{s_2}, \cdots y_{s_p})$ being taken between y_s and y_0, a finite number of segments fulfilling the requirements of the theorem. In case any segment extends beyond the given interval (α, β), we take only that portion of it which falls within (α, β).

The condition given by the theorem is also sufficient. To show this, let us suppose this condition fulfilled, and show that $f(x, y_0)$ is a continuous function for the interval $\alpha \leqq x \leqq \beta$. Let x_0 be any value of x within this interval and let y_s be any value chosen from the set (y) dense at y_0. If we assign to y_s any particular value, say y_{s_1}, then by hypothesis there exists a finite number of segments of lines lying between $y = y_{s_1}$ and $y = y_0$ fulfilling the condition of the theorem. Upon some one of these lines, say $y = y_{t_1}$, there exists for x_0 a neighborhood for which

$$|f(x, y_0) - f(x, y_{t_1})| < \sigma.$$

Again, if we put $y_s = y_{s_2}$, where y_{s_2} lies between y_{t_1} and y_0, there must exist a finite number of segments of lines lying between $y = y_{s_2}$ and $y = y_0$ likewise satisfying the condition of the theorem. Upon some one of these lines, say $y = y_{t_2}$, there exists for x_0 a neighborhood for which

$$|f(x, y_0) - f(x, y_{t_2})| < \sigma.$$

Continuing in the same manner, we obtain for x_0 an infinite succession of neighborhoods each point of which satisfies the inequality

$$|f(x, y_0) - f(x, y_{t_n})| < \sigma,$$

where $\underset{n \doteq \infty}{\mathrm{L}}\, y_{t_n} = y_0$. These successive neighborhoods may vary in extent with y_{t_n}, but from some point on they are all different from zero. The condition of theorem I is therefore satisfied for the point $x = x_0$ and consequently $f(x, y_0)$ is continuous at that point. But x_0 was any point of the given interval $\alpha \leqq x \leqq \beta$, and hence $f(x, y_0)$ is continuous throughout the interval. With this, the demonstration is completed.

§ 4. As we saw in §1, the representation of a function by means of an infinite series of continuous functions may be regarded as a special case of the problem considered in §§ 2, 3. Theorem I, giving the necessary and sufficient condition that the limiting function shall be continuous for the value $x = x_0$ becomes the following for the case of infinite series.

I′. *Given an infinite series*

$$u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots$$

whose terms are continuous functions of x *for the interval* $\alpha \leqq x \leqq \beta$, *and which converges for every value of* x *within the same interval. In order that the function* $f(x)$ *defined by this series be continuous at a point* x_0 *of the given interval, it is necessary and sufficient that for an arbitrarily small positive number* σ *and for every value of* n *greater than a sufficiently large number* m *there shall exist a neighborhood, which, however, may vary in extent with* n, *such that for every value of* x *within it we have*

$$| f(x) - S_n(x) | \equiv | R_n(x) | < \sigma,$$

where $S_n(x)$ *denotes the sum of the first* n *terms of the series and* $R_n(x)$ *is the remainder.*

From theorem II we obtain the necessary and sufficient condition that $f(x)$ shall be continuous throughout the given interval $\alpha \leqq x \leqq \beta$. This condition may be stated for the infinite series as follows:

II′. *Given an infinite series of continuous functions which converges toward a limit* $f(x)$ *for each value of* x *within a definite interval* $\alpha \leqq x \leqq \beta$. *In order that* $f(x)$ *shall be continuous throughout this interval, it is necessary and sufficient that for an arbitrarily small positive number* σ *and for any integer* m_1 *there shall exist another integer* $m_2 > m_1$ *such that for some integer* m *lying between* m_1 *and* m_2 *we have*

$$| f(x) - S_m(x) | \equiv | R_m(x) | < \sigma, \quad \alpha \leqq x \leqq \beta,$$

where, however, m *may change its value a finite number of times as* x *varies from* α *to* β.

When we regard x and $1/n$ as the rectangular coördinates of a point in a plane, the above theorem may be stated more clearly perhaps as follows, where we make the same assumptions as before concerning the nature of the terms of the infinite series and its convergence.

II''. *An arbitrarily small positive number* σ *and an integer* m *being chosen entirely at pleasure, then in order that* $f(x)$, *defined as above, shall be continuous throughout the interval* $\alpha \leqq x \leqq \beta$, *it is necessary and sufficient that there shall exist a finite number of segments of the lines*

$$n = m + p_1, \quad m + p_2, \cdots, \quad m + p_r$$

fulfilling the following conditions: The sum of the projections of these segments upon the x-axis shall fill the entire interval $\alpha \leqq x \leqq \beta$, *and every point* $(x, 1/n)$ *of these segments shall satisfy the inequality*

$$|f(x) - S_n(x)| \equiv |R_n(x)| < \sigma.$$

Theorems II', II'' introduce a kind of convergence of infinite series which Arzelà has called uniform convergence by segments (convergenza uniforme a tratti), although sub-uniform convergence seems a more appropriate name.* It differs from ordinary uniform convergence in that for uniform convergence each segment must fill the entire interval in question, and, moreover, such a segment must be present for every value of n greater than some given integer. As we have seen, neither of these conditions has to be fulfilled for the case of sub-uniform convergence. In fact, a series might converge sub-uniformly in a given interval and yet not be uniformly convergent in any sub-division of that interval, however small.

On the other hand, sub-uniform convergence differs from the simple uniform convergence (einfach gleichmässige Convergenz) introduced by Dini.† While Dini's simple uniform convergence is like sub-uniform convergence in that it does not require that the segments mentioned in the theorem shall be present for all values of n from a certain point on, it differs from sub-uniform convergence in requiring that these segments shall each fill the entire interval in question. From these considerations it follows that when we have uniform convergence of a series in the ordinary sense, or simple uniform convergence in the Dini sense, the series converges sub-uniformly in the Arzelà sense. The converse, however, is not true. Since sub-uniform convergence is the necessary and sufficient condition for the continuity of the limiting function, it follows from what has been said that any series which converges uniformly or simply uniformly defines a continuous function, provided the terms of the series are also continuous functions within the given closed interval.

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* See also Moore: BULLETIN, vol. 7, March, 1901, p. 257.

† See: Grundlagen, etc., p. 137.

The distinction between these various kinds of convergence is illustrated by the following examples.

1. Let $f(x)$ be defined by the series

$$1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \qquad 0 \leqq x \leqq \tfrac{1}{2}.$$

We have then

$$S_n(x) = \frac{x^n - 1}{x - 1}, \quad f(x) = \frac{1}{1 - x}, \quad |R_n(x)| = \frac{x^n}{x - 1}.$$

If now we select σ arbitrarily small, we can determine a value of n, say n_1, such that for $n > n_1$ we have

$$|R_n(x)| < \sigma$$

for all values of x for which the series is defined. In other words, if we consider any of the points on the lines given by putting

$$n = n_1, \quad n_1 + 1, \quad n_1 + 2, \cdots$$

the above inequality holds for all values of x within the given interval. The series converges uniformly therefore within this interval. We have also simple uniform convergence, since the projection of each of these segments upon the x-axis fills the entire interval $0 \leqq x \leqq \frac{1}{2}$ and the corresponding values of n from a set dense at infinity. Moreover, the series converges sub-uniformly; for it is at once evident that the conditions of theorem II'' are satisfied.

2. Given the series whose terms are formed in accordance with the following law:

$$u_{2m-1} = \frac{x}{mx^2 + (1 - mx)^2},$$

$$u_{2m} = \frac{-x}{(m + 1)x^2 + [1 - (m + 1)x]^2},$$

$$(m = 1, 2, 3, \cdots; \ -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}).$$

The series converges for all values of x within the given interval and gives

$$f(x) = \frac{x}{x^2 + 1 - x}.$$

When n is odd, say of the form $2m - 1$, we have

$$S_n(x) = \frac{x}{x^2 + (1 - x)^2}, \quad R_n(x) = 0.$$

Hence for all odd values of n, however large, we have

$$| R_n(x) | < \sigma.$$

On the other hand, when n is even, say of the form $2m - 2$, we have

$$S_n(x) = \frac{x}{x^2 + (1-x)^2} - \frac{x}{mx^2 + (1-mx)^2}, \; |R_n(x)| = \frac{x}{mx^2 + (1- mx)^2}$$

As n takes increasing even values, it is impossible to find any integer beyond which for all such values of n, and for all values of x within the given interval, the inequality

$$|R_n(x)| < \sigma$$

shall exist when we choose $\sigma < 1$. This follows from the fact that the values of $|R_n(x)|$ approach 1 along the line $x = 1/m$ as n approaches infinity. Hence for $\sigma < 1$, those segments which fill the entire interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$, and for each point of which the above inequality is valid, can exist only for odd values of n. These values of n are, however, dense at infinity. Hence we have simple uniform convergence, but not ordinary uniform convergence. However, the conditions of theorem II″ are satisfied and the series converges sub-uniformly.

3. Given the series, the sum of whose first n terms is

$$S_n(x) = \frac{nx}{1 + n^2x^2}, \quad -1 \leqq x \leqq +1$$

This series converges for each value of x within this interval and gives

$$f(x) = \underset{n \doteq \infty}{\mathrm{L}} S_n(x) = 0.$$

For any finite value of n, $S_n(x)$ at one point on the line $x = 1/n$ is equal to $\frac{1}{2}$ and at one point on the line $x = -1/n$ is equal to $-\frac{1}{2}$. Hence, for $\sigma < \frac{1}{2}$, there are no segments filling the entire interval $(-1, +1)$ such that for each point of them we have

$$|R_n(x)| < \sigma.$$

It follows that the series does not converge uniformly in the given interval, nor do we have simple uniform convergence. However, the series does converge sub-uniformly; for we may, for any arbitrary σ, select the required segments as follows.

First of all, consider the series for values of x in the neighborhood of $x = 0$. For $x = 0$, we have

$$f(0) = \underset{n=\infty}{\mathrm{L}} S_n(0) = 0.$$

Hence there exists a definite value of n, say $n = n_1$, such that

$$(1) \qquad |f(0) - S_{n_1}(0)| < \frac{\sigma}{3}.$$

Because $S_n(x)$ is continuous in x, there exists upon the line $n = n_1$ an interval $(0 - \delta_{n_1}, 0 + \delta'_{n_1})$ such that for every value of x within it, we have

$$(2) \qquad |S_{n_1}(0) - S_{n_1}(x)| < \frac{\sigma}{3}.$$

Moreover, since $f(x)$ equals zero for all values of x within the given interval $(-1, +1)$, we have for all values of x under consideration the inequality

$$(3) \qquad |f(x) - f(0)| < \frac{\sigma}{3}.$$

By combining the inequalities (1), (2), (3), we obtain the following relation

$$(4) \qquad |f(x) - S_1(x)| \equiv |R_{n_1}(x)| < \sigma,$$

which holds for all values of x within the interval $(0 - \delta_{n_1}, 0 + \delta'_{n_1})$ taken on the line $n = n_1$. Excluding this sub-interval from consideration, the given series converges uniformly throughout the remaining portion of the interval $(-1, +1)$. Hence, for the same value of σ, we can find on some line $n = n_2$, $n_2 > n_1$, a segment filling the entire remaining part of the given interval which lies to the right of $x = \delta_{n_1}$, such that at each point of this segment the inequality

$$| R_{n_2}(x) | < \sigma$$

is valid. For the same reason, we can find on some line $n = n_3$, $n_3 > n_1$, a segment filling the remaining part of the given interval to the left of the point $x = -\delta_{n_1}$, such that for each point of this segment the above inequality holds.

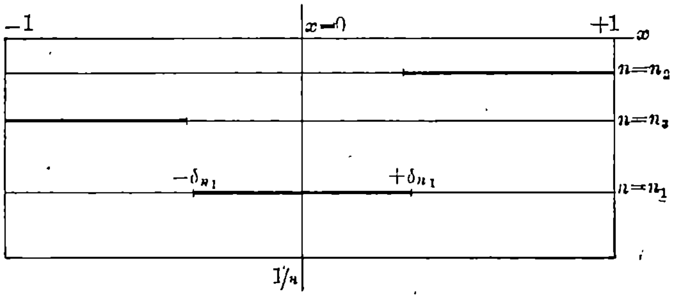

FIG. 4.

The three segments taken together satisfy the conditions of theorem II′, and hence the series converges sub-uniformly in the interval $-1 \leqq x \leqq 1$.

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GALOIS FIELD TABLES FOR $p^n \leqq 169$.

BY DR. W. H. BUSSEY.

(Read before the American Mathematical Society, September 7, 1905).

EVERY field of a finite number of marks may be represented as a Galois field of order $s = p^n$, where p^n is a power of a prime. The $GF[p^n]$ is defined uniquely by its order, and is therefore independent of the particular irreducible congruence used in its construction. In each of the following tables the $GF[p^n]$ is constructed by means of a primitive irreducible congruence which appears at the top of the table. The marks of each field are arranged in two tables. In each table each mark appears as a power of a primitive root i, and also as a polynomial in i of degree $k \leqq n - 1$. The coefficients in this polynomial are integers reduced modulo p. The mark $Ai^k + Bi^{k-1} + \cdots + Di + E, A \neq 0$ is denoted by $AB \cdots DE$, a symbol consisting of its detached coefficients in order. Zero coefficients must not be omitted. This is the usual symbol for a positive integer in the notation of the number system whose base is p. In the first table the marks are arranged according to ascending powers of i. In the second table the marks are arranged so that the symbols $AB \cdots DE$ represent the positive integers in natural order. By means of these two tables it is possible to perform with ease the operations of addition, subtraction, multiplication and division, within the field.

For an exposition of the Galois field theory, see Dickson's Linear Groups, pages 1–54; Jordan's Traité des Substitutions, pages 14–18, pages 156–161; Serret's Algèbre supérieure. For other references on Galois fields and higher irreducible congruences, see the preface to Dickson's Linear Groups.

Example 1. Simplify $(i^7 + i^{13})(i^2 + 3i + 4)$, i being a primitive root of the $GF[7^2]$.

From the first table for $GF[7^2]$, $i^7 = 6i + 1$, $i^{13} = 3i + 3$, $i^2 = i + 4$.

Therefore $i^7 + i^{13} = 9i + 4 = 2i + 4$ (modulo 7) $= i^6$ (by second table).

Also $i^2 + 3i + 4 = 4i + 8 = 4i + 1$ (modulo 7) $= i^{22}$ (by second table).

Therefore $(i^7 + i^{13})(i^2 + 3i + 4) = i^6 \cdot i^{22} = i^{28} = 5i + 1$ (by first table).

Example 2. Simplify $\dfrac{i^{11}+3i^{79}}{2i^{26}-3i^{8}+2}$, i being a primitive root of the $GF[5^3]$.

From the first table for $GF[5^3]$, $i^{11}=3i^2+3i$, $i^{79}=3i^2+i+1$, $i^{26}=3i+1$, $-i^8=i^{70}=3i^2+4i+4$.

Therefore $i^{11}+3i^{79}=3i^2+3i+3(3i^2+i+1)=12i^2+6i+3=2i+i+3 \pmod{5}=i^{81}$ (by second table).

Also $2i^{26}-3i^8+2=2(3i+1)+3(3i^2+4i+4)+2=9i^2+18i+16=4i^2+3i+1 \pmod{5}=i^{18}$ (by second table).

Therefore

$$\frac{i^{11}+i^{79}}{2i^{26}-3i^{8}+2}=\frac{i^{81}}{i^{18}}=i^{63}=4i \text{ (by first table).}$$

Example 3. Simplify $\dfrac{i^{88}-10i^{7}+4}{i^{83}+2/i^{117}}$, i being a primitive root of the $GF[11^2]$.

From the first table for $GF[11^2]$, $i^{88}=6i+2$, $i^7=9i+2$, $i^{83}=2i+3$, $i^3=3i+3$.

Also $-10i^7=i^7$, because $-10\equiv 1$ modulo 11; and $2/i^{117}=2i^3$ because $i^{120}=1$.

Therefore $i^{88}-10i^7+4=(6i+2)+(9i+2)+4=15i+8=4i+8$ (modulo 11) $=i^{82}$ (by second table).

Also $i^{83}+2/i^{117}=(2i+3)+2(3i+3)=8i+9=i^{101}$ (by second table).

Therefore

$$\frac{i^{88}-10i^{7}+4}{i^{83}+2/i^{117}}=\frac{i^{82}}{i^{101}}=\frac{i^{82}\cdot i^{120}}{i^{101}}=\frac{i^{202}}{i^{101}}=i^{101}=8i+9 \text{ (by first table).}$$

$GF[2^3]$, $i^3\equiv i+1$, modulo 2. $i^\lambda=\alpha i^2+\beta i+\gamma$.

FIRST TABLE.

λ	α	β	γ
1		1	0
2	1	0	0
3		1	1
4	1	1	0
5	1	1	1
6	1	0	1
7			1

SECOND TABLE.

λ	α	β	γ
7			1
1		1	0
3		1	1
2	1	0	0
6	1	0	1
4	1	1	0
5	1	1	1

$GF[3^2]$, $i^2 \equiv i+1$, modulo 3. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β
1	1	0	5	2	0
2	1	1	6	2	2
3	2	1	7	1	2
4		2	8		1

SECOND TABLE.

λ	α	β	λ	α	β
8		1	7	1	2
4		2	5	2	0
1	1	0	3	2	1
2	1	1	6	2	2

$GF[2^4]$, $i^4 \equiv i+1$, modulo 2. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.

λ	α	β	γ	δ
1			1	0
2		1	0	0
3	1	0	0	0
4			1	1
5		1	1	0
6	1	1	0	0
7	1	0	1	1
8		1	0	1
9	1	0	1	0
10		1	1	1
11	1	1	1	0
12	1	1	1	1
13	1	1	0	1
14	1	0	0	1
15				1

SECOND TABLE.

λ	α	β	γ	δ
15				1
1			1	0
4			1	1
2		1	0	0
8		1	0	1
5		1	1	0
10		1	1	1
3	1	0	0	0
14	1	0	0	1
9	1	0	1	0
7	1	0	1	1
6	1	1	0	0
13	1	1	0	1
11	1	1	1	0
12	1	1	1	1

$GF[5^2]$, $i^2 \equiv 2i+2$, modulo 5. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β
1	1	0	13	4	0
2	2	2	14	3	3
3	1	4	15	4	1
4	1	2	16	4	3
5	4	2	17	1	3
6		3	18		2
7	3	0	19	2	0
8	1	1	20	4	4
9	3	2	21	2	3
10	3	1	22	2	4
11	2	1	23	3	4
12		4	24		1

SECOND TABLE.

λ	α	β	λ	α	β
24		1	21	2	3
18		2	22	2	4
6		3	7	3	0
12		4	10	3	1
1	1	0	9	3	2
8	1	1	14	3	3
4	1	2	23	3	4
17	1	3	13	4	0
3	1	4	15	4	1
19	2	0	5	4	2
11	2	1	16	4	3
2	2	2	20	4	4

$GF[2^5]$, $i^5 \equiv i^3 + i^2 + i + 1$, modulo 2. $i^\lambda = \alpha i^4 + \beta i^3 + \gamma i^2 + \delta i + \epsilon$.

FIRST TABLE.

λ	α	β	γ	δ	ε
1				1	0
2			1	0	0
3		1	0	0	0
4	1	0	0	0	0
5		1	1	1	1
6	1	1	1	1	0
7	1	0	0	1	1
8		1	0	0	1
9	1	0	0	1	0
10		1	0	1	1
11	1	0	1	1	0
12				1	1
13			1	1	0
14		1	1	0	0
15	1	1	0	0	0
16	1	1	1	1	1
17	1	0	0	0	1
18		1	1	0	1
19	1	1	0	1	0
20	1	1	0	1	1
21	1	1	0	0	1
22	1	1	1	0	1
23	1	0	1	0	1
24			1	0	1
25		1	0	1	0
26	1	0	1	0	0
27			1	1	1
28		1	1	1	0
29	1	1	1	0	0
30	1	0	1	1	1
31					1

SECOND TABLE.

λ	α	β	γ	δ	ε
31					1
1				1	0
12				1	1
2			1	0	0
24			1	0	1
13			1	1	0
27			1	1	1
3		1	0	0	0
8		1	0	0	1
25		1	0	1	0
10		1	0	1	1
14		1	1	0	0
18		1	1	0	1
28		1	1	1	0
5		1	1	1	1
4	1	0	0	0	0
17	1	0	0	0	1
9	1	0	0	1	0
7	1	0	0	1	1
26	1	0	1	0	0
23	1	0	1	0	1
11	1	0	1	1	0
30	1	0	1	1	1
15	1	1	0	0	0
21	1	1	0	0	1
19	1	1	0	1	0
20	1	1	0	1	1
29	1	1	1	0	0
22	1	1	1	0	1
6	1	1	1	1	0
16	1	1	1	1	1

$GF[3^3]$, $i^3 \equiv i + 2$, modulo 3. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
1		1	0	10	1	1	0	19	2	2	2
2	1	0	0	11	1	1	2	20	2	1	1
3		1	2	12	1	0	2	21	1	0	1
4	1	2	0	13			2	22		2	2
5	2	1	2	14		2	0	23	2	2	0
6	1	1	1	15	2	0	0	24	2	2	1
7	1	2	2	16		2	1	25	2	0	1
8	2	0	2	17	2	1	0	26			1
9		1	1	18	1	2	1				

$GF[3^3]$, $i^\lambda = \alpha i^2 + \beta i + \gamma$.

SECOND TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
26			1	21	1	0	1	25	2	0	1
13			2	12	1	0	2	8	2	0	2
1		1	0	10	1	1	0	17	2	1	0
9		1	1	6	1	1	1	20	2	1	1
3		1	2	11	1	1	2	5	2	1	2
14		2	0	4	1	2	0	23	2	2	0
16		2	1	18	1	2	1	24	2	2	1
22		2	2	7	1	2	2	19	2	2	2
2	1	0	0	15	2	0	0				

$GF[7^2]$, $i^2 \equiv i + 4$, modulo 7. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β
1	1	0	25	6	0
2	1	4	26	6	3
3	5	4	27	2	3
4	2	6	28	5	1
5	1	1	29	6	6
6	2	4	30	5	3
7	6	1	31	1	6
8		3	32		4
9	3	0	33	4	0
10	3	5	34	4	2
11	1	5	35	6	2
12	6	4	36	1	3
13	3	3	37	4	4
14	6	5	38	1	2
15	4	3	39	3	4
16		2	40		5
17	2	0	41	5	0
18	2	1	42	5	6
19	3	1	43	4	6
20	4	5	44	3	2
21	2	2	45	5	5
22	4	1	46	3	6
23	5	2	47	2	5
24		6	48		1

SECOND TABLE.

λ	α	β	λ	α	β
48		1	39	3	4
16		2	10	3	5
8		3	46	3	6
32		4	33	4	0
40		5	22	4	1
24		6	34	4	2
1	1	0	15	4	3
5	1	1	37	4	4
38	1	2	20	4	5
36	1	3	43	4	6
2	1	4	41	5	0
11	1	5	28	5	1
31	1	6	23	5	2
17	2	0	30	5	3
18	2	1	3	5	4
21	2	2	45	5	5
27	2	3	42	5	6
6	2	4	25	6	0
47	2	5	7	6	1
4	2	6	35	6	2
9	3	0	26	6	3
19	3	1	12	6	4
44	3	2	14	6	5
13	3	3	29	6	6

$GF[2^6]$, $i^6 = i + 1$, modulo 2. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \varsigma$.

FIRST TABLE.

λ	α	β	γ	δ	ε	ς	λ	α	β	γ	δ	ε	ζ
1					1	0	33		1	0	0	1	0
2				1	0	0	34	1	0	0	1	0	0
3			1	0	0	0	35			1	0	1	1
4		1	0	0	0	0	36		1	0	1	1	0
5	1	0	0	0	0	0	37	1	0	1	1	0	0
6					1	1	38		1	1	0	1	1
7				1	1	0	39	1	1	0	1	1	0
8			1	1	0	0	40	1	0	1	1	1	1
9		1	1	0	0	0	41		1	1	1	0	1
10	1	1	0	0	0	0	42	1	1	1	0	1	0
11	1	0	0	0	1	1	43	1	1	0	1	1	1
12				1	0	1	44	1	0	1	1	0	1
13			1	0	1	0	45		1	1	0	0	1
14		1	0	1	0	0	46	1	1	0	0	1	0
15	1	0	1	0	0	0	47	1	0	0	1	1	1
16		1	0	0	1	1	48			1	1	0	1
17	1	0	0	1	1	0	49		1	1	0	1	0
18			1	1	1	1	50	1	1	0	1	0	0
19		1	1	1	1	0	51	1	0	1	0	1	1
20	1	1	1	1	0	0	52		1	0	1	0	1
21	1	1	1	0	1	1	53	1	0	1	0	1	0
22	1	1	0	1	0	1	54		1	0	1	1	1
23	1	0	1	0	0	1	55	1	0	1	1	1	0
24		1	0	0	0	1	56		1	1	1	1	1
25	1	0	0	0	1	0	57	1	1	1	1	1	0
26				1	1	1	58	1	1	1	1	1	1
27			1	1	1	0	59	1	1	1	1	0	1
28		1	1	1	0	0	60	1	1	1	0	0	1
29	1	1	1	0	0	0	61	1	1	0	0	0	1
30	1	1	0	0	1	1	62	1	0	0	0	0	1
31	1	0	0	1	0	1	63						1
32			1	0	0	1							

SECOND TABLE.

λ	α	β	γ	δ	ε	ζ	λ	α	β	γ	δ	ε	ζ
63						1	27			1	1	1	0
1					1	0	18			1	1	1	1
6					1	1	4		1	0	0	0	0
2				1	0	0	24		1	0	0	0	1
12				1	0	1	33		1	0	0	1	0
7				1	1	0	16		1	0	0	1	1
26				1	1	1	14		1	0	1	0	0
3			1	0	0	0	52		1	0	1	0	1
32			1	0	0	1	36		1	0	1	1	0
13			1	0	1	0	54		1	0	1	1	1
35			1	0	1	1	9		1	1	0	0	0
8			1	1	0	0	45		1	1	0	0	1
48			1	1	0	1	49		1	1	0	1	0

$GF[2^6]$, $i^6 \equiv i + 1$, modulo 2. $i^\lambda = \alpha i^5 + \beta i^4 + \gamma i^3 + \delta i^2 + \epsilon i + \zeta$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	ϵ	ζ	λ	α	β	γ	δ	ϵ	ζ
38		1	1	0	1	1	55	1	0	1	1	1	0
28		1	1	1	0	0	40	1	0	1	1	1	1
41		1	1	1	0	1	10	1	1	0	0	0	0
19		1	1	1	1	0	61	1	1	0	0	0	1
56		1	1	1	1	1	46	1	1	0	0	1	0
5	1	0	0	0	0	0	30	1	1	0	0	1	1
62	1	0	0	0	0	1	50	1	1	0	1	0	0
25	1	0	0	0	1	0	22	1	1	0	1	0	1
11	1	0	0	0	1	1	39	1	1	0	1	1	0
34	1	0	0	1	0	0	43	1	1	0	1	1	1
31	1	0	0	1	0	1	29	1	1	1	0	0	0
17	1	0	0	1	1	0	60	1	1	1	0	0	1
47	1	0	0	1	1	1	42	1	1	1	0	1	0
15	1	0	1	0	0	0	21	1	1	1	0	1	1
23	1	0	1	0	0	1	20	1	1	1	1	0	0
53	1	0	1	0	1	0	59	1	1	1	1	0	1
51	1	0	1	0	1	1	57	1	1	1	1	1	0
37	1	0	1	1	0	0	58	1	1	1	1	1	1
44	1	0	1	1	0	1							

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.

λ	α	β	γ	δ	λ	α	β	γ	δ
1			1	0	24		2	2	2
2		1	0	0	25	2	2	2	0
3	1	0	0	0	26			2	2
4	2	2	1	1	27		2	2	0
5		2	0	2	28	2	2	0	0
6	2	0	2	0	29		1	2	2
7	1	0	2	2	30	1	2	2	0
8	2	1	0	1	31	1	1	1	1
9	2	1	0	2	32			2	1
10	2	1	1	2	33		2	1	0
11	2	2	1	2	34	2	1	0	0
12		2	1	2	35	2	1	2	2
13	2	1	2	0	36	2	0	1	2
14	2	0	2	2	37	1	2	1	2
15	1	0	1	2	38	1	0	0	1
16	2	0	0	1	39	2	2	2	1
17	1	1	0	2	40				2
18		2	0	1	41			2	0
19	2	0	1	0	42		2	0	0
20	1	2	2	2	43	2	0	0	0
21	1	1	0	1	44	1	1	2	2
22		2	2	1	45		1	0	1
23	2	2	1	0	46	1	0	1	0

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3. $i^\lambda = \alpha i^3 + \beta i^2 + \gamma i + \delta$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	δ	λ	α	β	γ	δ
47	2	0	1	1	64		1	1	1
48	1	2	0	2	65	1	1	1	0
49	1	2	0	1	66			1	1
50	1	2	2	1	67		1	1	0
51	1	1	2	1	68	1	1	0	0
52		1	2	1	69		2	1	1
53	1	2	1	0	70	2	1	1	0
54	1	0	1	1	71	2	2	2	2
55	2	0	2	1	72			1	2
56	1	0	0	2	73		1	2	0
57	2	2	0	1	74	1	2	0	0
58		1	0	2	75	1	2	1	1
59	1	0	2	0	76	1	0	2	1
60	2	1	1	1	77	2	1	2	1
61	2	2	0	2	78	2	0	0	2
62		1	1	2	79	1	1	1	2
63	1	1	2	0	80				1

SECOND TABLE.

λ	α	β	γ	δ	λ	α	β	γ	δ
80				1	3	1	0	0	0
40				2	38	1	0	0	1
1			1	0	56	1	0	0	2
66			1	1	46	1	0	1	0
72			1	2	54	1	0	1	1
41			2	0	15	1	0	1	2
32			2	1	59	1	0	2	0
26			2	2	76	1	0	2	1
2		1	0	0	7	1	0	2	2
45		1	0	1	68	1	1	0	0
58		1	0	2	21	1	1	0	1
67		1	1	0	17	1	1	0	2
64		1	1	1	65	1	1	1	0
62		1	1	2	31	1	1	1	1
73		1	2	0	79	1	1	1	2
52		1	2	1	63	1	1	2	0
29		1	2	2	51	1	1	2	1
42		2	0	0	44	1	1	2	2
18		2	0	1	74	1	2	0	0
5		2	0	2	49	1	2	0	1
33		2	1	0	48	1	2	0	2
69		2	1	1	53	1	2	1	0
12		2	1	2	75	1	2	1	1
27		2	2	0	37	1	2	1	2
22		2	2	1	30	1	2	2	0
24		2	2	2					

$GF[3^4]$, $i^4 \equiv 2i^3 + 2i^2 + i + 1$, modulo 3.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	λ	α	β	γ	δ
50	1	2	2	1	60	2	1	1	1
20	1	2	2	2	10	2	1	1	2
43	2	0	0	0	13	2	1	2	0
16	2	0	0	1	77	2	1	2	1
78	2	0	0	2	35	2	1	2	2
19	2	0	1	0	28	2	2	0	0
47	2	0	1	1	57	2	2	0	1
36	2	0	1	2	61	2	2	0	2
6	2	0	2	0	23	2	2	1	0
55	2	0	2	1	4	2	2	1	1
14	2	0	2	2	11	2	2	1	2
34	2	1	0	0	25	2	2	2	0
8	2	1	0	1	39	2	2	2	1
9	2	1	0	2	71	2	2	2	2
70	2	1	1	0					

$GF[11^2]$, $i^2 \equiv 4i + 9$, modulo 11. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
1	1	0	31	3	8	61	10	0	91	8	3
2	4	9	32	9	5	62	7	2	92	2	6
3	3	3	33	8	4	63	8	8	93	3	7
4	4	5	34	3	6	64	7	6	94	8	5
5	10	3	35	7	5	65	1	8	95	4	6
6	10	2	36		8	66	1	9	96		3
7	9	2	37	8	0	67	2	9	97	3	0
8	5	4	38	10	6	68	6	7	98	1	5
9	2	1	39	2	2	69	9	10	99	9	9
10	9	7	40	10	7	70	2	4	100	1	4
11	10	4	41	3	2	71	1	7	101	8	9
12		2	42	3	5	72		9	102	8	6
13	2	0	43	6	5	73	9	0	103	5	6
14	8	7	44	7	10	74	3	4	104	4	1
15	6	6	45	5	8	75	5	5	105	6	3
16	8	10	46	6	1	76	3	1	106	5	10
17	9	6	47	3	10	77	2	5	107	8	1
18	9	4	48		5	78	2	7	108		6
19	7	4	49	5	0	79	4	7	109	6	0
20	10	8	50	9	1	80	1	3	110	2	10
21	4	2	51	4	4	81	7	9	111	7	7
22	7	3	52	9	3	82	4	8	112	2	8
23	9	8	53	6	4	83	2	3	113	5	7
24		4	54	6	10	84		7	114	5	1
25	4	0	55	1	10	85	7	0	115	10	1
26	5	3	56	3	9	86	6	8	116	8	2
27	1	1	57	10	5	87	10	10	117	1	6
28	5	9	58	1	2	88	6	2	118	10	9
29	7	1	59	6	9	89	4	10	119	5	2
30	7	8	60		10	90	4	3	120		1

$GF[11^2]$, $i^2 = 4i + 9$, modulo 11. $i^\lambda = \alpha i + \beta$.

SECOND TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
120		1	67	2	9	103	5	6	91	8	3
12		2	110	2	10	113	5	7	33	8	4
96		3	97	3	0	45	5	8	94	8	5
24		4	76	3	1	28	5	9	102	8	6
48		5	41	3	2	106	5	10	14	8	7
108		6	3	3	3	109	6	0	63	8	8
84		7	74	3	4	46	6	1	101	8	9
36		8	42	3	5	88	6	2	16	8	10
72		9	34	3	6	105	6	3	73	9	0
60		10	93	3	7	53	6	4	50	9	1
1	1	0	31	3	8	43	6	5	7	9	2
27	1	1	56	3	9	15	6	6	52	9	3
58	1	2	47	3	10	68	6	7	18	9	4
80	1	3	25	4	0	86	6	8	32	9	5
100	1	4	104	4	1	59	6	9	17	9	6
98	1	5	21	4	2	54	6	10	10	9	7
117	1	6	90	4	3	85	7	0	23	9	8
71	1	7	51	4	4	29	7	1	99	9	9
65	1	8	4	4	5	62	7	2	69	9	10
66	1	9	95	4	6	22	7	3	61	10	0
55	1	10	79	4	7	19	7	4	115	10	1
13	2	0	82	4	8	35	7	5	6	10	2
9	2	1	2	4	9	64	7	6	5	10	3
39	2	2	89	4	10	111	7	7	11	10	4
83	2	3	49	5	0	30	7	8	57	10	5
70	2	4	114	5	1	81	7	9	38	10	6
77	2	5	119	5	2	44	7	10	40	10	7
92	2	6	26	5	3	37	8	0	20	10	8
78	2	7	8	5	4	107	8	1	118	10	9
112	2	8	75	5	5	116	8	2	87	10	10

$GF[5^3]$, $i^3 = 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
1		1	0	13	1	0	4	25	2	0	4
2	1	0	0	14		1	3	26		3	1
3		2	3	15	1	3	0	27	3	1	0
4	2	3	0	16	3	2	3	28	1	1	4
5	3	4	1	17	2	4	4	29	1	1	3
6	4	2	4	18	4	3	1	30	1	0	3
7	2	2	2	19	3	4	2	31			3
8	2	1	1	20	4	3	4	32		3	0
9	1	0	1	21	3	2	2	33	3	0	0
10		3	3	22	2	3	4	34		1	4
11	3	3	0	23	3	3	1	35	1	4	0
12	3	1	4	24	3	2	4	36	4	2	3

$GF[5^3]$, $i^3 \equiv 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
37	2	1	2	67	2	1	4	97	4	1	0
38	1	1	1	68	1	3	1	98	1	3	2
39	1	3	3	69	3	3	3	99	3	4	3
40	3	0	3	70	3	4	4	100	4	4	4
41		4	4	71	4	0	4	101	4	2	2
42	4	4	0	72		2	2	102	2	0	2
43	4	3	2	73	2	2	0	103		1	1
44	3	0	2	74	2	4	1	104	1	1	0
45		3	4	75	4	0	1	105	1	2	3
46	3	4	0	76		4	2	106	2	0	3
47	4	1	4	77	4	2	0	107		2	1
48	1	2	2	78	2	3	2	108	2	1	0
49	2	4	3	79	3	1	1	109	1	4	1
50	4	2	1	80	1	2	4	110	4	3	3
51	2	4	2	81	2	1	3	111	3	1	2
52	4	1	1	82	1	2	1	112	1	3	4
53	1	4	2	83	2	3	3	113	3	1	3
54	4	4	3	84	3	2	1	114	1	4	4
55	4	1	2	85	2	2	4	115	4	1	3
56	1	0	2	86	2	3	1	116	1	1	2
57		4	3	87	3	0	1	117	1	4	3
58	4	3	0	88		2	4	118	4	0	3
59	3	3	2	89	2	4	0	119		1	2
60	3	3	4	90	4	4	1	120	1	2	0
61	3	0	4	91	4	4	2	121	2	2	3
62			4	92	4	0	2	122	2	2	1
63		4	0	93			2	123	2	0	1
64	4	0	0	94		2	0	124			1
65		3	2	95	2	0	0				
66	3	2	0	96		4	1				

SECOND TABLE.

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
124			1	45		3	4	48	1	2	2
93			2	63		4	0	105	1	2	3
31			3	96		4	1	80	1	2	4
62			4	76		4	2	15	1	3	0
1		1	0	57		4	3	68	1	3	1
103		1	1	41		4	4	98	1	3	2
119		1	2	2	1	0	0	39	1	3	3
14		1	3	9	1	0	1	112	1	3	4
34		1	4	56	1	0	2	35	1	4	0
94		2	0	30	1	0	3	109	1	4	1
107		2	1	13	1	0	4	53	1	4	2
72		2	2	104	1	1	0	117	1	4	3
3		2	3	38	1	1	1	114	1	4	4
88		2	4	116	1	1	2	95	2	0	0
32		3	0	29	1	1	3	123	2	0	1
26		3	1	28	1	1	4	102	2	0	2
65		3	2	120	1	2	0	106	2	0	3
10		3	3	82	1	2	1	25	2	0	4

$GF[5^3]$, $i^3 \equiv 2i + 3$, modulo 5. $i^\lambda = \alpha i^2 + \beta i + \gamma$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	λ	α	β	γ	λ	α	β	γ
108	2	1	0	61	3	0	4	118	4	0	3
8	2	1	1	27	3	1	0	71	4	0	4
37	2	1	2	79	3	1	1	97	4	1	0
81	2	1	3	111	3	1	2	52	4	1	1
67	2	1	4	113	3	1	3	55	4	1	2
73	2	2	0	12	3	1	4	115	4	1	3
122	2	2	1	66	3	2	0	47	4	1	4
7	2	2	2	84	3	2	1	77	4	2	0
121	2	2	3	21	3	2	2	50	4	2	1
85	2	2	4	16	3	2	3	101	4	2	2
4	2	3	0	24	3	2	4	36	4	2	3
86	2	3	1	11	3	3	0	6	4	2	4
78	2	3	2	23	3	3	1	58	4	3	0
83	2	3	3	59	3	3	2	18	4	3	1
22	2	3	4	69	3	3	3	43	4	3	2
89	2	4	0	60	3	3	4	110	4	3	3
74	2	4	1	46	3	4	0	20	4	3	4
51	2	4	2	5	3	4	1	42	4	4	0
49	2	4	3	19	3	4	2	90	4	4	1
17	2	4	4	99	3	4	3	91	4	4	2
33	3	0	0	70	3	4	4	54	4	4	3
87	3	0	1	64	4	0	0	100	4	4	4
44	3	0	2	75	4	0	1				
40	3	0	3	92	4	0	2				

$GF[2^7]$, $i^7 \equiv i + 1$, modulo 2. $i^{-\lambda} = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \epsilon i^2 + \zeta i + \eta$.

FIRST TABLE.

λ	α	β	γ	δ	ε	ζ	η	λ	α	β	γ	δ	ε	ζ	η
1						1	0	24	1	1	1	1	0	0	0
2					1	0	0	25	1	1	1	0	0	1	1
3				1	0	0	0	26	1	1	0	0	1	0	1
4			1	0	0	0	0	27	1	0	0	1	0	0	1
5		1	0	0	0	0	0	28			1	0	0	0	1
6	1	0	0	0	0	0	0	29		1	0	0	0	1	0
7						1	1	30	1	0	0	0	1	0	0
8					1	1	0	31				1	0	1	1
9				1	1	0	0	32			1	0	1	1	0
10			1	1	0	0	0	33		1	0	1	1	0	0
11		1	1	0	0	0	0	34	1	0	1	1	0	0	0
12	1	1	0	0	0	0	0	35		1	1	0	0	1	1
13	1	0	0	0	0	1	1	36	1	1	0	0	1	1	0
14					1	0	1	37	1	0	0	1	1	1	1
15				1	0	1	0	38			1	1	1	0	1
16			1	0	1	0	0	39		1	1	1	0	1	0
17		1	0	1	0	0	0	40	1	1	1	0	1	0	0
18	1	0	1	0	0	0	0	41	1	1	0	1	0	1	1
19		1	0	0	0	1	1	42	1	0	1	0	1	0	1
20	1	0	0	0	1	1	0	43		1	0	1	0	0	1
21				1	1	1	1	44	1	0	1	0	0	1	0
22			1	1	1	1	0	45		1	0	0	1	1	1
23		1	1	1	1	0	0	46	1	0	0	1	1	1	0

$GF[2^7]$, $i^7 = i + 1$, modulo 2. $i^{-\lambda} = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \varepsilon i^2 + \zeta i + \eta$.

FIRST TABLE.—*Continued.*

λ	α	β	γ	δ	ϵ	ζ	η	λ	α	β	γ	δ	ϵ	ζ	η
47			1	1	1	1	1	88	1	1	0	0	0	1	0
48		1	1	1	1	1	0	89	1	0	0	0	1	1	1
49	1	1	1	1	1	0	0	90				1	1	0	1
50	1	1	1	1	0	1	1	91			1	1	0	1	0
51	1	1	1	0	1	0	1	92		1	1	0	1	0	0
52	1	1	0	1	0	0	1	93	1	1	0	1	0	0	0
53	1	0	1	0	0	0	1	94	1	0	1	0	0	1	1
54		1	0	0	0	0	1	95		1	0	0	1	0	1
55	1	0	0	0	0	1	0	96	1	0	0	1	0	1	0
56					1	1	1	97			1	0	1	1	1
57				1	1	1	0	98		1	0	1	1	1	0
58			1	1	1	0	0	99	1	0	1	1	1	0	0
59		1	1	1	0	0	0	100		1	1	1	0	1	1
60	1	1	1	0	0	0	0	101	1	1	1	0	1	1	0
61	1	1	0	0	0	1	1	102	1	1	0	1	1	1	1
62	1	0	0	0	1	0	1	103	1	0	1	1	1	0	1
63				1	0	0	1	104		1	1	1	0	0	1
64			1	0	0	1	0	105	1	1	1	0	0	1	0
65		1	0	0	1	0	0	106	1	1	0	0	1	1	1
66	1	0	0	1	0	0	0	107	1	0	0	1	1	0	1
67			1	0	0	1	1	108			1	1	0	0	1
68		1	0	0	1	1	0	109		1	1	0	0	1	0
69	1	0	0	1	1	0	0	110	1	1	0	0	1	0	0
70			1	1	0	1	1	111	1	0	0	1	0	1	1
71		1	1	0	1	1	0	112			1	0	1	0	1
72	1	1	0	1	1	0	0	113		1	0	1	0	1	0
73	1	0	1	1	0	1	1	114	1	0	1	0	1	0	0
74		1	1	0	1	0	1	115		1	0	1	0	1	1
75	1	1	0	1	0	1	0	116	1	0	1	0	1	1	0
76	1	0	1	0	1	1	1	117		1	0	1	1	1	1
77		1	0	1	1	0	1	118	1	0	1	1	1	1	0
78	1	0	1	1	0	1	0	119		1	1	1	1	1	1
79		1	1	0	1	1	1	120	1	1	1	1	1	1	0
80	1	1	0	1	1	1	0	121	1	1	1	1	1	1	1
81	1	0	1	1	1	1	1	122	1	1	1	1	1	0	1
82		1	1	1	1	0	1	123	1	1	1	1	0	0	1
83	1	1	1	1	0	1	0	124	1	1	1	0	0	0	1
84	1	1	1	0	1	1	1	125	1	1	0	0	0	0	1
85	1	1	0	1	1	0	1	126	1	0	0	0	0	0	1
86	1	0	1	1	0	0	1	127							1
87		1	1	0	0	0	1								

$GF[2^7]$, $i^7 = i+1$, modulo 2. $i^\lambda = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \epsilon i^2 + \zeta i + \eta$.

SECOND TABLE.

λ	α	β	γ	δ	ϵ	ζ	η	λ	α	β	γ	δ	ϵ	ζ	η
127							1	87		1	1	0	0	0	1
1						1	0	109		1	1	0	0	1	0
7						1	1	35		1	1	0	0	1	1
2					1	0	0	92		1	1	0	1	0	0
14					1	0	1	74		1	1	0	1	0	1
8					1	1	0	71		1	1	0	1	1	0
56					1	1	1	79		1	1	0	1	1	1
3				1	0	0	0	59		1	1	1	0	0	0
63				1	0	0	1	104		1	1	1	0	0	1
15				1	0	1	0	39		1	1	1	0	1	0
31				1	0	1	1	100		1	1	1	0	1	1
9				1	1	0	0	23		1	1	1	1	0	0
90				1	1	0	1	82		1	1	1	1	0	1
57				1	1	1	0	48		1	1	1	1	1	0
21				1	1	1	1	119		1	1	1	1	1	1
4			1	0	0	0	0	6	1	0	0	0	0	0	0
28			1	0	0	0	1	126	1	0	0	0	0	0	1
64			1	0	0	1	0	55	1	0	0	0	0	1	0
67			1	0	0	1	1	13	1	0	0	0	0	1	1
16			1	0	1	0	0	30	1	0	0	0	1	0	0
112			1	0	1	0	1	62	1	0	0	0	1	0	1
32			1	0	1	1	0	20	1	0	0	0	1	1	0
97			1	0	1	1	1	89	1	0	0	0	1	1	1
10			1	1	0	0	0	66	1	0	0	1	0	0	0
108			1	1	0	0	1	27	1	0	0	1	0	0	1
91			1	1	0	1	0	96	1	0	0	1	0	1	0
70			1	1	0	1	1	111	1	0	0	1	0	1	1
58			1	1	1	0	0	69	1	0	0	1	1	0	0
38			1	1	1	0	1	107	1	0	0	1	1	0	1
22			1	1	1	1	0	46	1	0	0	1	1	1	0
47			1	1	1	1	1	37	1	0	0	1	1	1	1
5		1	0	0	0	0	0	18	1	0	1	0	0	0	0
54		1	0	0	0	0	1	53	1	0	1	0	0	0	1
29		1	0	0	0	1	0	44	1	0	1	0	0	1	0
19		1	0	0	0	1	1	94	1	0	1	0	0	1	1
65		1	0	0	1	0	0	114	1	0	1	0	1	0	0
95		1	0	0	1	0	1	42	1	0	1	0	1	0	1
68		1	0	0	1	1	0	116	1	0	1	0	1	1	0
45		1	0	0	1	1	1	76	1	0	1	0	1	1	1
17		1	0	1	0	0	0	34	1	0	1	1	0	0	0
43		1	0	1	0	0	1	86	1	0	1	1	0	0	1
113		1	0	1	0	1	0	78	1	0	1	1	0	1	0
115		1	0	1	0	1	1	73	1	0	1	1	0	1	1
33		1	0	1	1	0	0	99	1	0	1	1	1	0	0
77		1	0	1	1	0	1	103	1	0	1	1	1	0	1
98		1	0	1	1	1	0	118	1	0	1	1	1	1	0
117		1	0	1	1	1	1	81	1	0	1	1	1	1	1
11		1	1	0	0	0	0	12	1	1	0	0	0	0	0

$GF[2^7]$, $i^7 \equiv i+1$, modulo 2. $i^\lambda = \alpha i^6 + \beta i^5 + \gamma i^4 + \delta i^3 + \epsilon i^2 + \zeta i + \eta$.

SECOND TABLE.—*Continued.*

λ	α	β	γ	δ	ϵ	ζ	η	λ	α	β	γ	δ	ϵ	ζ	η
125	1	1	0	0	0	0	1	124	1	1	1	0	0	0	1
88	1	1	0	0	0	1	0	105	1	1	1	0	0	1	0
61	1	1	0	0	0	1	1	25	1	1	1	0	0	1	1
110	1	1	0	0	1	0	0	40	1	1	1	0	1	0	0
26	1	1	0	0	1	0	1	51	1	1	1	0	1	0	1
36	1	1	0	0	1	1	0	101	1	1	1	0	1	1	0
106	1	1	0	0	1	1	1	84	1	1	1	0	1	1	1
93	1	1	0	1	0	0	0	24	1	1	1	1	0	0	0
52	1	1	0	1	0	0	1	123	1	1	1	1	0	0	1
75	1	1	0	1	0	1	0	83	1	1	1	1	0	1	0
41	1	1	0	1	0	1	1	50	1	1	1	1	0	1	1
72	1	1	0	1	1	0	0	49	1	1	1	1	1	0	0
85	1	1	0	1	1	0	1	122	1	1	1	1	1	0	1
80	1	1	0	1	1	1	0	120	1	1	1	1	1	1	0
102	1	1	0	1	1	1	1	121	1	1	1	1	1	1	1
60	1	1	1	0	0	0	0								

$GF[13^2]$, $i^2 \equiv i+11$, modulo 13. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
1	1	0	25	7	5	49	4	11	73	7	1
2	1	11	26	12	12	50	2	5	74	8	12
3	12	11	27	11	2	51	7	9	75	7	10
4	10	2	28		4	52	3	12	76	4	12
5	12	6	29	4	0	53	2	7	77	3	5
6	5	2	30	4	5	54	9	9	78	8	7
7	7	3	31	9	5	55	5	8	79	2	10
8	10	12	32	1	8	56		3	80	12	9
9	9	6	33	9	11	57	3	0	81	8	2
10	2	8	34	7	8	58	3	7	82	10	10
11	10	9	35	2	12	59	10	7	83	7	6
12	6	6	36	1	9	60	4	6	84		12
13	12	1	37	10	11	61	10	5	85	12	0
14		2	38	8	6	62	2	6	86	12	2
15	2	0	39	1	10	63	8	9	87	1	2
16	2	9	40	11	11	64	4	10	88	3	11
17	11	9	41	9	4	65	1	5	89	1	7
18	7	4	42		8	66	6	11	90	8	11
19	11	12	43	8	0	67	4	1	91	6	10
20	10	4	44	8	10	68	5	5	92	3	1
21	1	6	45	5	10	69	10	3	93	4	7
22	7	11	46	2	3	70		6	94	11	5
23	5	12	47	5	9	71	6	0	95	3	4
24	4	3	48	1	3	72	6	1	96	7	7

$GF[13^2]$, $i^2 = i + 11$, modulo 13. $i^\lambda = \alpha i + \beta$.

FIRST TABLE.—*Continued.*

λ	α	β	λ	α	β	λ	α	β	λ	α	β
97	1	12	115	4	8	133	9	2	151	9	12
98		11	116	12	5	134	11	8	152	8	8
99	11	0	117	4	2	135	6	4	153	3	10
100	11	4	118	6	5	136	10	1	154		7
101	2	4	119	11	1	137	11	6	155	7	0
102	6	9	120	12	4	138	4	4	156	7	12
103	2	1	121	3	2	139	8	5	157	6	12
104	3	9	122	5	7	140		10	158	5	1
105	12	7	123	12	3	141	10	0	159	6	3
106	6	2	124	2	2	142	10	6	160	9	1
107	8	1	125	4	9	143	3	6	161	10	8
108	9	10	126		5	144	9	7	162	5	6
109	6	8	127	5	0	145	3	8	163	11	3
110	1	1	128	5	3	146	11	7	164	1	4
111	2	11	129	8	3	147	5	4	165	5	11
112		9	130	11	10	148	9	3	166	3	3
113	9	0	131	8	4	149	12	8	167	6	7
114	9	8	132	12	10	150	7	2	168		1

SECOND TABLE.

λ	α	β	λ	α	β	λ	α	β	λ	α	β
168		1	15	2	0	52	3	12	165	5	11
14		2	103	2	1	29	4	0	23	5	12
56		3	124	2	2	67	4	1	71	6	0
28		4	46	2	3	117	4	2	72	6	1
126		5	101	2	4	24	4	3	106	6	2
70		6	50	2	5	138	4	4	159	6	3
154		7	62	2	6	30	4	5	135	6	4
42		8	53	2	7	60	4	6	118	6	5
112		9	10	2	8	93	4	7	12	6	6
140		10	16	2	9	115	4	8	167	6	7
98		11	79	2	10	125	4	9	109	6	8
84		12	111	2	11	64	4	10	102	6	9
1	1	0	35	2	12	49	4	11	91	6	10
110	1	1	57	3	0	76	4	12	66	6	11
87	1	2	92	3	1	127	5	0	157	6	12
48	1	3	121	3	2	158	5	1	155	7	0
164	1	4	166	3	3	6	5	2	73	7	1
65	1	5	95	3	4	128	5	3	150	7	2
21	1	6	77	3	5	147	5	4	7	7	3
89	1	7	143	3	6	68	5	5	18	7	4
22	1	8	58	3	7	162	5	6	25	7	5
36	1	9	145	3	8	122	5	7	83	7	6
39	1	10	104	3	9	55	5	8	96	7	7
2	1	11	153	3	10	47	5	9	34	7	8
97	1	12	88	3	11	45	5	10	51	7	9

$GF[13^2]$, $i^2 \equiv i + 11$, modulo 13. $i^\lambda = \alpha i + \beta$.

SECOND TABLE.—*Continued.*

λ	α	β	λ	α	β	λ	α	β	λ	α	β
75	7	10	160	9	1	61	10	5	17	11	9
22	7	11	133	9	2	142	10	6	130	11	10
156	7	12	148	9	3	59	10	7	40	11	11
43	8	0	41	9	4	161	10	8	19	11	12
107	8	1	31	9	5	11	10	9	85	12	0
81	8	2	9	9	6	82	10	10	13	12	1
129	8	3	144	9	7	37	10	11	86	12	2
131	8	4	114	9	8	8	10	12	123	12	3
139	8	5	54	9	9	99	11	0	120	12	4
38	8	6	108	9	10	119	11	1	116	12	5
78	8	7	33	9	11	27	11	2	5	12	6
152	8	8	151	9	12	163	11	3	105	12	7
63	8	9	141	10	0	100	11	4	149	12	8
44	8	10	136	10	1	94	11	5	80	12	9
90	8	11	4	10	2	137	11	6	132	12	10
74	8	12	69	10	3	146	11	7	3	12	11
113	9	0	20	10	4	134	11	8	26	12	12

NOTES.

THE July number (volume 6, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Sur les lignes géodésiques des surfaces convexes," by H. POINCARÉ; "The classification of quadrics," by T. J. I'A. BROMWICH; "On differential invariants," by J. E. WRIGHT; "Groups of order p^m, which contain cyclic subgroups of order p^{m-3}," by L. I. NEIKIRK; "On the invariant subgroups of prime index," by G. A. MILLER; "On a general method for treating transmitted motions and its application to indirect perturbations," by E. W. BROWN; "On hypercomplex number systems," by L. E. DICKSON; "A theorem on finite algebras," by J. H. MACLAGAN-WEDDERBURN; "The relation of the principles of logic to the foundations of geometry," by J. ROYCE; "On multiple integrals," by J. PIERPONT.

THE July number (volume 27, number 3) of the *American Journal of Mathematics* contains: "Deduction of the power series representing a function from special values of the latter," by G. W. HILL; "On the definition of reducible hypercomplex number systems," by S. EPSTEEN and H. B. LEONARD;

"Quintic curves for which $p = 1$," by P. FIELD; "Classification of the surfaces of singularities of the quadratic spherical complex," by C. L. E. MOORE; "Subgroups of order a power of p in the general and special m-ary linear homogeneous groups in the $GF[p^n]$," by L. E. DICKSON.

THE concluding (July) number of volume 6 of the *Annals of Mathematics* contains: "On the real elements of certain classes of geometrical configurations," by L. E. DICKSON; "The continuum as a type of order; an exposition of the modern theory, I–IV," by E. V. HUNTINGTON; "On integrating factors," by P. SAUREL; "Concerning series of analytic functions," by M. B. PORTER.

THE conference of associations of teachers of mathematics and physics, held at Asbury Park on July 5 in connection with the meeting of the National educational association was attended by thirty-seven delegates representing nearly all the larger teachers' associations in this field. Professor THOMAS S. FISKE was elected chairman, and Dr. ARTHUR SCHULTZE secretary of the conference. It was decided to organize a national society of teachers of mathematics and science and a committee was appointed to arrange the details of the organization.

WHILE in this country last autumn Professor G. Darboux became interested in the proper reporting of articles appearing in American mathematical publications for the *Bulletin des sciences mathématiques*. Arrangements have been made by which the *American Journal of Mathematics*, the *Transactions of the American Mathematical Society* and the *Annals of Mathematics* will be regularly reviewed in Darboux's *Bulletin* by Professor J. W. YOUNG, of Princeton University, Professor E. R. HEDRICK, of the University of Missouri, and Dr. C. M. MASON, of Yale University, respectively. Articles of an original nature appearing in the BULLETIN will also be mentioned. Authors of articles in these publications will render valuable service by furnishing abstracts of their articles to the respective reviewers.

THE following books will shortly appear from the G. J. Göschen'sche publishing house, Leipzig: "Auslese aus meiner Unterrichts- und Vorlesungspraxis," by Dr. H. SCHUBERT; "Vorlesungen über photographische Optik," by Dr. A. GLEICHEN; "Zwölf Vorlesungen über die Natur des Lichtes," by Dr. J. CLASSEN; "Mehrdimensionale Geometrie, II. die Polytope (Sammlung Schubert, number 36)," by Professor P. H. SCHOUTE.

UNIVERSITY OF COLORADO. — The following advanced courses in mathematics are announced for the academic year 1905–1906: By Professor I. M. DELONG: Theory of equations, five hours; Differential equations, three hours. — By Dr. S. EPSTEEN: Linear algebra, three hours; Least squares and theory of errors, three hours; Mathematics of investments (including insurance), three hours. — By Mr. H. B. LEONARD: Theory of Functions, three hours. — By Miss R. L. CARSTENS: History of mathematics, two hours.

THE various foreign universities below offer courses in mathematics for the winter semester of 1905–1906 as follows:

UNIVERSITY OF GOTTINGEN. — By Professor F. KLEIN: Projective geometry, including non-euclidean geometry, four hours; Seminar, two hours. — By Professor D. HILBERT: Introduction to partial differential equations, two hours; Mechanics, four hours; Seminar, two hours. — By Professor H. MINKOWSKI: Algebraic curves and surfaces, two hours; Theory of numbers, four hours; Seminar (with Professor Hilbert), two hours. — By Professor C. RUNGE: Differential and integral calculus, II, four hours; Graphical methods in mechanics, three hours. — By Professor K. SCHWARZSCHILD: Celestial mechanics, II, three hours. — By Professor W. VOIGT: Elasticity and acoustics, four hours; Electro-optics, two hours. — By Dr. E. ZERMELO: Functions of real variables, three hours. — By Dr. M. ABRAHAM: Maxwell's theory and Hertzian waves, three hours. — By Dr. O. BLUMENTHAL: Theory of surfaces, three hours. — By Dr. C. HERGLOTZ: Minimum surfaces, two hours. — By Dr. C. CARATHÉODORY: Continuous groups, three hours.

UNIVERSITY OF HALLE. — By Professor G. CANTOR: Theory of analytic functions, four hours; Seminar, two hours. — By Professor A. WANGERIN: Applications of elliptic functions, two hours; Theory of potential and spherical harmonics, four hours; Seminar, two hours. — By Professor A. GUTZMER: Analytic mechanics, four hours; Theory and applications of determinants, two hours; Seminar, two hours. — By Professor V. EBERHARD: Integral calculus, four hours; with exercises, one hour. — By Dr. F. BERNSTEIN : Analytic geometry of space, two hours ; Theory of differential equations, two hours.

UNIVERSITY OF INNSBRUCK. — By Professor O. STOLZ : Differential and integral calculus of real variables, three hours ; Concept of complex numbers as introduction to the theory of functions, two hours ; Seminar, one hour. — By Professor K. ZINDLER : Analytic geometry of two and three dimensions, with exercises, seven hours. — By Professor K. MENGER : Descriptive geometry, three hours.

UNIVERSITY OF JENA. — By Professor J. THOMAE : Elementary theory of functions, four hours ; Analytic geometry of space, four hours ; Seminar, two hours. — By ——— (not yet appointed) : Integral calculus with exercises, five hours ; Theory and applications of determinants, two hours ; Elements of the theory of numbers, two hours. — By Professor G. FREGE : Analytic mechanics, four hours ; Seminar, two hours.

UNIVERSITY OF KÖNIGSBERG. — By Professor W. F. MEYER : Theory of algebraic equations, four hours ; Seminar, one hour. — By Professor A. SCHOENFLIES : Integral calculus, four hours ; with exercises, two hours. — By Professor L. SAALSCHÜTZ : Introduction to algebraic analysis, four hours ; Geometry of space, three hours ; with exercises, one hour. — By Dr. I. COHN : Theory of potential, three hours.

UNIVERSITY OF MUNICH. — By Professor G. BAUER : Seminar, two hours. — By Professor F. LINDEMANN : Differential calculus, five hours ; Analytic mechanics, four hours ; Seminar, one and one half hours. — By Professor A. VOSS : Plane analytic geometry, five hours ; Algebraic surfaces, four hours ; Seminar, two hours. — By Professor A. PRINGSHEIM : Elliptic functions, four hours ; Continued fractions, two hours. — By Professor K. DOEHLEMANN : Descriptive geometry, I, five hours ; with exercises, three hours ; Synthetic geometry, five hours ; with exercises, one hour. — By Professor E. v. WEBER : Algebra, four hours ; Integral calculus with exercises, five hours. — By Dr. H. BRUNN : Theory of aggregates, four hours.

OXFORD UNIVERSITY. *Michaelmas term.* — By Professor W. ESSON : Analytic geometry of plane curves, two hours ;

Synthetic geometry of plane curves, one hour. — By Professor E. B. ELLIOTT: Sequences and series, two hours; Elementary theory of numbers, one hour. — By Professor A. E. H. LOVE: Mathematical theory of electricity and magnetism, three hours. — By Professor H. H. Turner: Elementary mathematical astronomy, two hours. — By Mr. C. E. HASELFOOT: Theory of equations, one hour. — By Mr. C. LEUDESDORF: Projective geometry, three hours. — By Mr. A. E. JOLLIFFE: Analytic geometry, two hours. — By Mr. J. W. RUSSELL: Differential calculus, two hours. — By Mr. B. F. McNEILE: Curve tracing, one hour. — By Mr. A. L. PEDDER: Problems in pure mathematics, one hour. — By Mr. C. H. SAMPSON: Solid geometry, two hours. — By Mr. J. E. CAMPBELL: Differential equations, two hours. — By Mr. C. H. THOMPSON: Integral calculus, two hours.

UNIVERSITY OF WÜRZBURG. — By Professor F. PRYM: Differential calculus with introduction to higher analysis, four hours; Plane analytic geometry, I, four hours; Proseminar, two hours; Seminar, two hours. — By Professor E. SELLING: Theory of algebraic equations, four hours; Analytic mechanics, four hours. — By Professor G. ROST: Theory of partial differential equations, four hours; Theory of invariants, four hours; Analytic geometry of space, four hours; Proseminar, six hours; Seminar, two hours.

THE Academy of sciences of Berlin held its Leibniz session on June 29. The Steiner prize was not awarded, but the sum of six thousand marks was set apart in recognition of the investigations of the late Professor GUIDO HAUCK. For the year 1910 the Academy announces the same problem for the third time for the Steiner prize:

"To solve completely any important, hitherto unsolved problem relating to the theory of curved surfaces, taking into account, so far as possible, the methods and principles evolved by Steiner. It is required that sufficient analytic explanations shall accompany the geometric investigations to verify the correctness and completeness of the solution. Without wishing to limit the choice of subject, the academy takes the opportunity to call attention to the special problems to which Steiner has referred in his general remark at the end of his second paper on maximum and minimum in figures in a plane, on a sphere, and in space."

Competing memoirs should be written in German, French, English, Italian or Latin, and must be submitted before December 31, 1909, under the usual conditions. The result will be announced at the Leibniz session of 1910.

THE London mathematical society has awarded the De Morgan prize for the present year to Dr. H. F. BAKER, for his work in analysis.

THE Hungarian academy of sciences has established a new prize, known as the Bolyai prize, in commemoration of JOHN and WOLFGANG BOLYAI. It consists of a medal and a sum of ten thousand crowns. It will be awarded this fall to the author of the best mathematical production during the last five years, and at intervals of five years thereafter. The committee of award consists of Professors Darboux, Klein, König and Rados; it will meet in Budapest in October.

PROFESSOR W. WIRTINGER has been elected a member of the academy of sciences of Vienna.

PROFESSOR J. FRANEL, of the technical school at Zürich, has been chosen director of that institution.

MR. E. T. WHITTAKER, of Trinity College, Cambridge, has been appointed university lecturer in mathematics.

PROFESSOR J. E. PAPPERITZ, of the mining academy at Freiberg, has been elected rector of the academy for the present academic year.

PROFESSOR C. NEUMANN, of the University of Leipzig, has been decorated with the red cross of the second class.

PROFESSOR W. VOIGT, of the University of Göttingen, has been decorated with the order of the red cross of the third class.

DR. H. C. TIMERDING has been appointed professor of applied mathematics at the University of Strassburg.

PROFESSOR A. GUTZMER, of the University of Jena, has been appointed professor of mathematics at the University of Halle.

PROFESSOR L. HEFFTER, of the technical school at Aachen, has been appointed professor of mathematics at the University of Kiel.

DR. O. BLUMENTHAL, of the University of Göttingen, has been appointed professor of mathematics at the technical school at Aachen.

DR. L. FEJER has been appointed docent in analysis and mechanics at the University of Clausenburg.

THE following changes have been made at the University of Illinois: Professor E. J. TOWNSEND has been made full professor of mathematics and acting dean of the college of science; Professor S. E. SLOCUM, of the University of Cincinnati, has been appointed to an assistant professorship of mathematics; Dr. L. NEIKIRK, of the University of Pennsylvania, has been appointed instructor in mathematics, and Mr. W. J. RISLEY, of the Armour Institute, and M. C. EMMENS, of Albion College, have been made assistants in mathematics. Professor A. G. HALL has been appointed professor of mathematics in Miami University, and Mr. A. H. WILSON has been appointed associate professor of mathematics at the Polytechnic Institute of Alabama.

THE College entrance examination board has appointed as examiners in mathematics for 1905–1906 Professor F. N. COLE, Professor H. S. WHITE, and Dr. ARTHUR SCHULTZE.

DR. D. R. CURTISS, of Yale University, has been appointed to an assistant professorship of mathematics at Northwestern University.

AT the University of Missouri, Dr. O. D. KELLOGG of Princeton University has been appointed to an assistant professorship of mathematics, and Mr. R. L. BORGER, of the University of Florida, to an instructorship in mathematics.

AT Northwestern University, Professor THOMAS F. HOLGATE has been appointed Noyes professor of pure mathematics; Mr. R. E. WILSON has returned from Europe to resume his work as instructor in mathematics; Dr. J. C. MOREHEAD has been appointed instructor in mathematics.

AT Princeton University Dr. L. P. EISENHART and Dr. W. GILLESPIE have been promoted to assistant professorships of mathematics. Dr. O. VEBLEN, of the University of Chicago, and Dr. J. W. YOUNG, of Northwestern University, have been appointed assistant professors of mathematics, and Dr. O. L. UNDERHILL instructor in mathematics.

DR. L. I. HEWES has been appointed instructor in mathematics at Yale University.

DR. R. B. ALLEN has been appointed lecturer in mathematics at Clark University during Professor Taber's absence.

DR. S. EPSTEEN, of the University of Chicago, has been appointed instructor in applied mathematics at the University of Colorado.

PROFESSOR A. EMCH, of the University of Colorado, has been appointed professor of mathematics at the cantonal college of Solothurn, Switzerland.

DR. O. P. AKERS has been appointed assistant professor of mathematics at Allegheny College, Meadville, Pennsylvania.

RECENT catalogues of second-hand mathematical books: Macmillan and Bowes, 1 Trinity St., Cambridge, England, catalogue no. 305, 5300 mathematical pamphlets; catalogue no. 302, from the libraries of Pendlebury, Spottiswoode and Whitehead, about 1400 titles in mathematics. —Martin Boas, Karltrasse 25, Berlin, catalogue no. 49, 56 titles in mathematics and geodesy. — H. R. Mecklenburg, 36 Klosterstrasse, Berlin, 750 titles in mathematics and geodesy.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Lezioni di geometria proiettiva dettata nella r. Università di Napoli. 3a edizione. Napoli, Pierro, 1905. 8vo. 14 + 456 pp.

BACHMANN (P.). Zahlentheorie. Versuch einer Gesamtdarstellung dieser Wissenschaft in ihren Hauptteilen. Band V: Die allgemeine Arithmetik der Zahlenkörper. Leipzig, Teubner, 1905. 8vo. 22 + 548 pp. M. 16.00

BERKHAU (G.). Zur projektiven Behandlung der Dreiecksgeometrie. (Diss.) Königsberg, 1905. 8vo. 31 pp.

BOCCHETTA (G.). Principi di geometria metrica e differenziale negli iperspazi. Melfi, Grieco, 1905. 8vo. 26 pp.

CAIN (W.). A brief course in the calculus. New York, Van Nostrand, 1905. 8vo. 10 + 280 pp. Cloth. (Van Nostrand series of text-books.) $1.75

COMBÉBIAC (G.). Calcul des triquaternions; nouvelle analyse géométrique. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 122 pp.

CREPCOEUR (A. J. M.). Cours d'analyse. Partie II: Calcul des différences et calcul des variations. Paris, 1905. 8vo. Fr. 3.50

Dölle (R.). Orthogonale Invarianten der Circularkurven dritter Ordnung. (Diss.) Jena, 1905. 8vo. 38 pp.

Epsteen (S.) and Leonard (H. B.). On the definition of reducible hypercomplex number systems. 4to. (*American Journal of Mathematics* 27, pp. 217–242).

Field (P.). Quintic curves for which $p = 1$. 4to. (*American Journal of Mathematics* 27, pp. 243–247, 10 plates.)

Fuhrmann (W.). Aufgaben aus der analytischen Geometrie. Königsberg, 1905. 4to. 28 pp.

Gerlich (P.). Ueber den Tangentenkomplex der Fläche zweiten Grades. (Diss.) Breslau, 1905. 8vo. 59 pp.

Giudice (M. del). Sulle equazioni algebriche di grado $2n$ la cui risoluzione si riconduce alla risoluzione, mediante equazioni di grado n, a quella di n equazioni quadratiche. Nota I e II. Napoli, Trani, 1904. 4to. 16 + 16 pp.

Halsted (G. B.). See Hilbert (D.).

Hermite (C.). Oeuvres, publiées sous les auspices de l'Académie des sciences par E. Picard. Vol. I. Paris, Gauthier-Villars, 1905. 8vo. 40 + 498 pp., portrait. Fr. 18.00

Hilbert (D.). On the foundations of logic and arithmetic. Translated by G. B. Halsted. 8vo. (*The Monist*, July, 1905, pp. 338–352.)

Krazer (A.). See Verhandlungen.

Lange (D. de). Eenige beschouwingen over enkel- en meervoudige ombilicaalpunten en den loop der krumtelijnen in hunne nabijheid. Amsterdam, 1904. 8vo. 8 + 82 pp. M. 3.00

Leonard (H. B.). See Epsteen (S.).

Lindow (M.). See Sohncke (L. A.).

Montessus de Ballore (R. de). Les fractions continues algébriques. (Thèse.) Paris, Hermann, 1905. 4to. 85 pp.

Müller (C. H.). Studien zur Geschichte der Mathematik, insbesondere des mathematischen Unterrichts an der Universität Göttingen im 18. Jahrhundert. Mit einer Einleitung: Ueber Charakter und Umfang historischer Forschung in der Mathematik. (Diss., Göttingen.) Leipzig, Teubner, 1904. 8vo. 94 pp.

Neumann (E. R.). Studien über die Methoden von C. Neumann und G. Robin zur Lösung der beiden Randwertaufgaben der Potentialtheorie. Leipzig, Teubner, 1905. 8vo. 23 + 194 pp. (Preisschriften, gekrönt und herausgegeben von der Fürstlich Jablonowskischen Gesellschaft zu Leipzig, 37, No. 15 der mathematisch-naturwissenschaftlichen Sektion.) M. 10.00

Picard (E.). See Hermite (C.).

Quidde (W.). Ueber Gauss'sche Kreise auf Rotationsflächen. (Diss.) Kiel, 1905. 8vo. 85 pp.

Salvadori (M.). Esposizione della teoria delle somme di Gauss e di alcuni teoremi di Eisenstein. (Tesi, Friburgo.) Pisa, 1905. 8vo. 116 pp.

Schmall (S. M.). A first course in analytical geometry, plane and solid, with numerous examples. New York, Van Nostrand, 1905. 8vo. 7 + 318 pp. Cloth. (Van Nostrand series of text-books.) $1.75

Sohncke (L. A.). Sammlung von Aufgaben aus der Differential- und Integralrechnung. Teil II: Integralrechnung. Abteilung 1. 6te, verbesserte Auflage, bearbeitet und herausgegeben von M. Lindow. Jena, Schmidt, 1905. 8vo. 12 + 221 pp. M. 4.00

Souchon (A.). Sur l'intégration des équations générales de la dynamique analytique et sur les principales propriétés de leurs intégrales canoniques. Tours, Deslis, 1905. 8vo. 23 pp.

Taber (H.). On hypercomplex number systems. 4to. (*Transactions of the American Mathematical Society*, 5, pp. 509–548.)

——— The scalar functions of hypercomplex numbers. 8vo. (*Proceedings of the American Academy of Arts and Sciences*, 41, pp. 59–70.)

Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8. bis 13. August 1904. Herausgegeben von dem Schriftführer des Kongresses A. Krazer. Mit einer Ansicht von Heidelberg in Heliogravüre und 2 Doppeltafeln. Leipzig, Teubner, 1905. 8vo. 10 + 756 pp. M. 18.00

Veronese (G.). La geometria non archimedea; una questione di priorità. Roma, 1905. 8vo. (*Rendiconti della r. Accademia dei Lincei* 14, pp. 247–251.)

Wieleitner (H.). Bibliographie der höheren algebraischen Kurven für den Zeitabschnitt von 1890–1904. Leipzig, 1904. 8vo. 53 pp. M. 1.50

Wildervanck (J. C.). De verschillende krommingen eener gewrongen kromme der vierdimensionale ruimte. Groningen, 1904. 8vo. 62 pp. M. 2.50

Wilson (E. B.). On products in additive fields. 8vo. (*Verhandlungen des III. Internationalen Mathematiker-Kongresses*, Heidelberg, 1904, pp. 202–215.)

II. ELEMENTARY MATHEMATICS.

Amodeo (F.). Equazioni di primo grado ad una incognita. Appendice all' Aritmetica particolare e generale estratta dagli Elementi di algebra dello stesso autore, per uso delle scuole normali e tecniche. Napoli, Pierro, 1904. 16mo. 31 pp. Fr. 0.30

Arbes (J.). Vierstellige Logarithmen-Tafeln zum Schulgebrauche. Wien, Tempsky, 1905. 8vo. 36 pp. Boards. M. 0.70

Bachet (C. G.). Problèmes plaisants et delectables qui se font par les nombres, par Claude-Gaspar Bachet, sieur de Méziriac. 4e édition, revue et simplifiée. Paris, Gauthier-Villars, 1905. 16mo. 170 pp. Fr. 3.50

Ball (W. W. R.). Mathematical recreations and essays. 4th edition. London, Macmillan, 1905. 12mo. 418 pp. Cloth. 7s.

Barnes (A.). See Consterdine (A.).

Borchardt (W. G.). Key to New trigonometry for schools. Part I. London, Bell, 1905. 12mo. Cloth. 5s.

Bourlet (C.). Précis d'algèbre, contenant cinq cent cinquante-sept exercices et problèmes rédigés conformément aux programmes du 31 mai 1902. Classes de seconde et première C et D. 2e édition. Paris, Hachette, 1905. 16mo. 393 pp. (Cours complet de mathématiques.) Fr. 2.50

Consterdine (A.) and Barnes (A.). Rudiments of practical mathematics. London, Murray, 1905. 12mo. 348 pp. Cloth. (School Library.) 2s. 6d.

CORNEJO (H.). See GRAIÑO (F.).

CURZON (H. S. J.). Permutations and combinations made easy. London, Simpkin, 1905. 12mo. (Normal tutorial series.) 6 d.

EHRIG (G.). Geometrie für die Zwecke des praktischen Lebens. Teil I: Geometrie der Ebene. Leipzig, Leineweber, 1905. 8vo. 8 + 121 pp. Cloth. M. 2.50

——. Geometrie für Baugewerkenschulen und verwandte technische und gewerbliche Lehranstalten, mit besonderer Berücksichtigung der praktischen Anwendungen. Teil I: Geometrie der Ebene. Leipzig, Leineweber, 1905. 8vo. 8 + 121 pp. Cloth. M. 2.50

F. J. Complément des "Eléments de géométrie," comprenant un complément sur le déplacement des figures. 9e édition. Paris, Poussielgue, [1905]. 16mo. Pp. 495–525.

FENKNER (H.). Arithmetische Aufgaben. Unter besonderer Berücksichtigung der Anwendungen aus dem Gebiete der Geometrie, Physik und Chemie. Für den Unterricht an höheren Lehranstalten bearbeitet. Ausgabe A: Vornehmlich für den Gebrauch in Gymnasien, Realgymnasien und Oberrealschulen. Teil I: Pensum der Untertertia, Obertertia und Untersekunda. 5te, verbesserte Auflage. Berlin, Salle, 1905. 8vo. 8 + 256 pp. M. 2.20

——. Ausgabe B: Vornehmlich für den Gebrauch in sechsklassigen höheren Lehranstalten und in Mittelschulen sowie in Seminaren und gewerblichen Fachschulen. 3te, verbesserte Auflage. Berlin, Salle, 1905. 8vo. 6 + 226 pp. M. 1.65

FERVAL (H.). Eléments de trigonométrie rédigés conformément aux programmes de l'enseignement secondaire et de l'enseignement primaire supérieur, contenant cinq cent cinquante-six exercices et problèmes.

Deuxième cycle (classes de seconde et de première C, D, et de mathématiques A, B). Paris, Hachette, 1905. 16mo. 288 pp. (Cours complet de mathématiques, par Bourlet et Ferval.) Fr. 2.50

FUCINI (C.). Geometria piana complementare e geometria solida per gl'istituti nautici. 7a edizione. Genova, 1905. 8vo. 125 pp. Fr. 1.80

——. Geometria piana e solida per gl'istituti nautici. 7a edizione. Genova, 1905. 8vo. 237 pp. Fr. 2.50

GEISS (G.). Raumlehre für höhere Mädchenschulen und Lehrerinnenseminare. Methodisch bearbeitet. 4te Auflage, herausgegeben und neu bearbeitet von K. Hensing. Bensheim, Ehrhard, 1905. 8vo. 6 + 178 pp. Cloth. M. 2.00

GODT (W.). Ueber einige sogenannte merkwürdige Punkte des Dreiecks. (Progr.) Lübeck, 1903. 4to. 15 pp.

GRAIÑO (F.), CORNEJO (H.), HERRERO (L.) y RIBERA (L. DE). Tablas de logaritmos, con cinco cifras decimales, seguida de una tabla de cuadrados de los números desde 1.000 hasta 9.999. Ferrol, Imp. de *El Correo Gallego*, 1905. 8vo. 144 pp. Cloth. Fr. 7.00

GUTSCHE (O.). Mathematische Uebungsaufgaben für Primaner von Realanstalten und jüngere Studierende gestellt und zum Teil mit Lösungen versehen. Leipzig, Teubner, 1905. 8vo. 4 + 82 pp. Boards. M. 1.20

HABERLAND (M.). Beziehungen der merkwürdigen Punkte eines Dreiecks zu den Ankreismittelpunktedreiecken, Potenzpunktedreiecken und Gegenpunktedreiecken. Neustrelitz, Barnewitz, 1905. 8vo. 20 pp. M. 0.50

Hall (H. S.) and Stevens (F. H.). Key to the exercises and examples contained in School geometry. Parts 1–4. London, Macmillan, 1905. 12mo. Cloth. 6s.

Hensing (K.). See Geiss (G.).

Herrero (L.). See Graiño (F.).

Holtze (A.). See Köstler (H).

Hoüel (G. J.). Fünfstellige Logarithmentafeln der Zahlen und der trigonometrischen Funktionen, nebst den Gaussischen Additions- und Subtraktionslogarithmen und verschiedenen Hülfstafeln. Neue, durchgesehene und vermehrte Ausgabe. Berlin, Prausnitz, 1905. 8vo. 46 + 118 pp. Boards. M. 2.50

Knott (C. G.). Four-figure mathematical tables. London, Chambers, 1905. 8vo. 24 pp. Cloth. 6d.

Köstler (H.). Leitfaden der ebenen Geometrie für höhere Lehranstalten, neu herausgegeben von A. Holtze. 2tes Heft: Lehre vom Flächeninhalt, Konstruktionslehre. 4te, teilweise umgearbeitete Auflage. Halle, Nebert, 1905. 8vo. 49 pp. Boards. M. 0.90

Krebs (A.). Allgemeine Arithmetik in elementarer Darstellung, für den Mittelschul- und Selbstunterricht in entwickelnder Lehrform bearbeitet. Heft I. Bern, Baumgart, 1905. 8vo. 3 + 79 pp. M. 1.40

Kutnewsky (M.). See Müller (H.).

Mackay (J. S.). Plane geometry, practical and geometrical. Books 1–5. London, Chambers, 1905. 12mo. 388 pp. Cloth. 3s. 6d.

Marsh (W. R.). Elementary algebra. New York, Scribner, 1905. 12mo. 7 + 395 pp. Cloth. (Marsh and Ashton mathematical series.) $1.00

Marx (E.). Ueber winkelhalbierende Linien des Dreiecks. (Progr.) Friedland, 1905. 4to. 17 pp.

Millis (J. F.). See Stone (J. C.).

Müller (H.) und Kutnewsky (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Ausgabe A: Für Gymnasien und Progymnasien. Teil I. 3te Auflage. Leipzig, Teubner, 1905. 8vo. 7 + 237 pp. Cloth. M. 2.20

Murray (D. A.). Plane trigonometry for colleges and secondary schools. [Also] Spherical trigonometry for colleges and secondary schools. New York, Longmans, 1905. 12mo. 13 + 212 + 9 + 95 pp. Cloth. $1.20

—— Practical mathematics, including solution of triangles, measurement of areas, heights and distances, the use of logarithms, plotting of graphs, and finding the slopes of curves, with four-place tables. New York, Longmans, 1905. 8vo. 10 + 113 pp. Cloth. $0.75

Payne (E. L.). Payne's natural methods in algebra; for district schools, high schools, and normal schools. Topeka, Kan., Crane, 1905. 12mo. 11 + 176 pp. Cloth. $0.60

Porchon (P.). Leçons de mathématiques conformes aux programmes du 31 mai 1902, pour les classes de philosophie A et B et pour les futurs élèves du P. C. N. Paris, Alcan, 1905. 16mo. 8 + 308 pp. Cloth. Fr. 3.50

Rankin (A. R.). An argument in symbols illustrating algebraic factoring by explanations of problems. Chicago, Flanagan Company, [1905]. 12mo. 24 pp. $0.15

REX (F. W.). Vierstellige Logarithmen-Tafeln. Schul-Ausgabe. 2te Auflage. Stuttgart, Metzler, 1905. 8vo. 2 + 29 pp. M. 0.60

RIBERA (L. DE). See GRAIÑO (F.).

SAXELBY (F. M.). A course in practical mathematics. London, Longmans, 1905. 8vo. 450 pp. Cloth. 6s. 6d.

SHORT (R. L.). Supplementary algebra. Boston, Heath, 1905. 12mo. 45 pp. (Heath's mathematical monographs.) $0.10

SHORTO (D. E.). Algebraical grounding. London, Rivingtons, 1905. 12mo. 1s.

SIEVERT (H.). Lehrbuch der Elementar-Geometrie zum Gebrauche an Mittelschulen und beim Selbstunterrichte. Teil I: Geometrie der Ebene. Abteilung I: Kongruenz, Gleichheit und Aehnlichkeit ebener Figuren. 2te Auflage. Leipzig, Deichert, 1905. 8vo. 6 + 179 pp. M. 2.60

STEVENS (F. H.). See HALL (H. S.).

STONE (J. C.) and MILLIS (J. F.). Essentials of algebra; complete course: an adequate preparation for the college or technical school; for secondary schools. Boston, Sanborn, 1905. 12mo. 12 + 412 pp. Half-leather. $1.12

THIEME (H.). Leitfaden der Mathematik für Gymnasien. Teil I: Die Unterstufe. 2te Auflage. Leipzig, Freytag, 1905. 8vo. 102 pp. Cloth. M. 1.60

——. Leitfaden der Mathematik für Realanstalten. Teil I: Die Unterstufe. 2te Auflage. Leipzig, Freytag, 1905. 8vo. 128 pp. Cloth. M. 1.60

TURNBULL (V. M.). Elementary plane geometry. London, Blackie, 1905. 12mo. 136 pp. Cloth. 2s.

WELL (G. J. VAN DE). Oplossingen der wiskundige opgaven van examen B der Polytechnische School te Delft; met nieuwe opgaven. Deel I: Vraagstukken over B1. 2te druk. Deventer, 1905. 8vo. 185 pp. M. 6.50

WILSON (E. B.). Seven lectures on spherical geometry. Reprinted from *The American Mathematical Monthly*, 1904. [Yale Coöperative Association.] 8vo. 34 pp. $0.35

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III. APPLIED MATHEMATICS.

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THE TWELFTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Twelfth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at Williams College, Williamstown, Mass., on Thursday and Friday, September 7–8, 1905. The following twenty-eight members of the Society were in attendance:

Professor G. A. Bliss, Professor W. G. Bullard, Dr. W. H. Bussey, Professor A. S. Chessin, Professor F. N. Cole, Professor L. L. Conant, Professor D. R. Curtiss, Professor E. W. Davis, Professor W. P. Durfee, Professor F. C. Ferry, Professor W. B. Fite, Professor A. S. Gale, Professor J. G. Hardy, Professor E. V. Huntington, Professor J. I. Hutchinson, Dr. Edward Kasner, Professor Frank Morley, Professor E. D. Roe, Dr. F. H. Safford, Miss M. E. Sinclair, Dr. C. H. Sisam, Professor Virgil Snyder, Professor W. E. Story, Professor H. W. Tyler, Professor H. S. White, Professor T. W. D. Worthen, Professor J. W. A. Young, Professor J. W. Young.

Two sessions were held on Thursday, and a third session on Friday morning. Professor Frank Morley and Professor F. C. Ferry were elected chairmen. The Council announced the election of the following persons to membership in the Society: Lieutenant Colonel C. P. Echols, U. S. Military Academy; Professor G. B. Guccia, University of Palermo; Professor H. B. Evans, University of Pennsylvania; Dr. A. M. Hiltebeitel, Princeton University; Dr. J. M. Poor, Dartmouth College; Professor J. E. Williams, Virginia Polytechnic Institute. Eight applications for membership in the Society were received.

It was decided to hold the annual meeting of the Society in New York on Thursday and Friday, December 28–29. A committee, consisting of Professors E. H. Moore, James Pierpont, and G. A. Miller, was appointed to prepare and report at the October meeting a list of nominations of officers and other members of the Council to be voted for at the annual meeting.

At the close of the Thursday morning session the members were conducted through the grounds and buildings of Williams College and the collection of mathematical models was shown. On Friday afternoon the members assembled at the house of

President Hopkins and through the courtesy of the college were taken in carriages on an excursion into the Berlin mountains whose less accessible regions were traversed on foot. Several foot tours were also made on Saturday. The hospitality of the college authorities was appropriately recognized by appreciative resolutions at the close of the meeting.

The following papers were read at the meeting:

(1) Dr. W. H. Bussey: "Galois field tables for $p^n \leqq 169$."

(2) Dr. Edward Kasner: "A geometric property of the trajectories of dynamics."

(3) Professor G. A. Bliss: "A generalization of the notion of angle."

(4) Professor W. B. Fite: "Irreducible linear homogeneous groups."

(5) Dr. Saul Epsteen: "Note on the structure of hypercomplex number systems."

(6) M. Maurice Fréchet: "Sur l'écart de deux courbes et sur les courbes limites."

(7) Mr. Richard Morris: "On the expressibility of the automorphic functions of the group $(0, 3; l_1, l_2, l_3)$ in terms of theta series."

(8) Professor J. I. Hutchinson: "On certain hyperabelian functions which are expressible by theta series."

(9) Mr. N. J. Lennes: "Concerning real functions of one real variable which are completely determined over an interval by the values of the function and its derivatives for one value of the independent variable."

(10) Dr. W. A. Manning: "On the arithmetic nature of the coefficients in groups of finite monomial linear substitutions."

(11) Dr. Max Mason: "On the boundary value problems of linear ordinary differential equations of the second order."

(12) Professor G. A. Miller: "On the possible number of operators of order 2 in a group of order 2^m."

(13) Professor Frank Morley: "On two cubic curves in triangular relation."

(14) Dr. C. H. Sisam: "On the determination of the nodal curve on a ruled surface."

(15) Professor A. S. Chessin: "On the strains and stresses in a rapidly revolving circular disc."

(16) Professor L. E. Dickson: "On the quaternary linear homogeneous groups modulo p of order a multiple of p."

(17) Professor L. E. DICKSON: "On finite algebras."

(18) Professor VIRGIL SNYDER: "On a type of rational twisted curves."

(19) Professor E. J. TOWNSEND: "Arzelà's condition for the continuity of a function defined by a series of continuous functions."

(20) Professor H. S. WHITE: "Rational plane curves as related to Riemann transformations."

(21) Professor F. R. MOULTON: "A class of periodic solutions of the problem of three bodies."

(22) Dr. C. N. HASKINS: "Note on the differential invariants of a surface and of space."

(23) Professor E. V. HUNTINGTON: "The continuum as a type of order: an exposition of the modern theory."

M. Fréchet's paper was communicated to the Society through Professor E. H. Moore, Mr. Morris's paper through Professor Hutchinson. In the absence of the authors, Mr. Morris's paper was read by Professor Hutchinson, Dr. Haskins's paper by Professor Bliss, and the papers of Dr. Epsteen, M. Fréchet, Mr. Lennes, Dr. Manning, Dr. Mason, Professor Miller, Professor Dickson, Professor Townsend, and Professor Moulton were read by title.

The papers of Dr. Bussey and Professor Townsend appeared in the October BULLETIN. Those of Dr. Kasner, Dr. Epsteen, Dr. Manning, and Professor Miller are included in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

3. Geometry has to do with a set of elements called points, which is divided into subsets called lines. If it is postulated that the elements of the original set can be put into one to one correspondence with the real number ratios $x : y : z$, then the points having ratios of the form $x : y : 1$ can be represented geometrically in the ordinary cartesian plane. A curve may be defined as the set of points corresponding to two equations $x = \phi(t)$, $y = \psi(t)$, and the length of a curve as the value of a definite integral of the form

$$I = \int F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) dt.$$

The nature of the geometry depends then upon the integral chosen to represent length. Hamel has used this notion in his determination of all geometries in which straight lines are the shortest distances, but without giving a corresponding generalization of angle. In Professor Bliss's paper such a generalization is developed, together with its application to surface theory and the so-called non-euclidean geometries.

4. It is well known that the number of variables in an irreducible linear homogeneous group is a divisor of the order of the group. In Professor Fite's paper this number is determined more closely for certain groups, and is definitely fixed for metabelian groups. A necessary and sufficient condition that a group whose order is a power of a prime or a group that is the direct product of such groups be simply isomorphic with an irreducible group is derived. These results for linear groups lead to certain properties of abstract groups. Finally the class of an irreducible group of order p^m in p variables (p a prime) is determined. This paper forms part of an article that is to appear in the *Transactions* under the title "Groups whose orders are powers of a prime."

6. The chief result of M. Fréchet's paper is to define a number called interspace (écart) which has the same properties in the theory of curves as distance in the theory of points: 1°. Two continuous curves C_1, C_2 being given, their interspace is a number, $(C_1, C_2) \geqq 0$, uniquely determined. 2°. The necessary and sufficient condition that two continuous curves coincide is that their interspace vanish. 3°. Given any three continuous curves C_1, C_2, C_3, then $(C_1, C_2) \leqq (C_1, C_3) + (C_3, C_2)$. Finally the concept of interspace is introduced into the theory of limiting curves.

7. Mr. Morris has studied the effect of a monodromy of the branch points of the Riemann surface $u^\nu = (z-a)^\alpha(z-b)^\beta (z-c)^\gamma(z-d)^\delta$, where $\alpha + \beta + \gamma + \delta \equiv 0$ (mod. ν), on the moduli of periodicity of the integrals of the first kind. If the surface be of genus p, and if $2n$ denote the number of linearly independent integrals which are not of lower genus, it is found that the number of linearly independent parameters in terms of which the moduli of these integrals are linearly expressible is in general not less than n. Under special condi-

tions on α, β, γ, δ the minimum possible value for this number reduces to $n/2$, or $n/4$. An important consequence is that the automorphic functions of class $(0, 3; l_1, l_2, l_3)$ cannot be expressed in terms of theta series containing only one variable in any cases other than those already known.

8. In the paper by Professor Hutchinson it is shown how the moduli of the theta functions can be specialized so as to be linearly expressible in terms of a certain number of independent parameters $\zeta_1, \zeta_2, \cdots$. Certain groups of transformations on these parameters correspond to linear transformations of the theta functions and thus lead to hyperabelian functions of $\zeta_1, \zeta_2, \cdots$, which are expressible in terms of theta series.

9. In Mr. Lennes's paper the following theorem is proved: If all the derivatives of a function $f(x)$ exist for every point of a segment $a \cdots b$ then a necessary and sufficient condition that $f(x)$ shall be expressible by means of Taylor's series at every point of the segment $a \cdots b$ is that every point x_k of this segment shall lie on a segment σ_k such that for some fixed number $k_k \neq 0$

$$\mathop{\mathrm{L}}_{n=\infty} \frac{f^{(n)}(x)k_k^n}{n!} = 0$$

uniformly for x on σ_k.

This condition differs essentially from that given by Pringsheim, though the proof in some respects is similar.

It is then shown that, if for a certain point the Taylor series is divergent or is convergent but fails to represent the function, then the function is not uniquely determined on any interval by the value of the function and those of its derivatives at that point, consequently it is impossible to devise an algorithm in terms of $f^{(i)}(x_0)$ $(i = 0, \cdots, \infty)$ which shall represent the function over an interval containing x_0 in case Taylor's expansion fails.

As a lemma a necessary and sufficient condition is given that a series of continuous functions whose sum is a continuous function shall be uniformly convergent.

11. Dr. Mason considered the general boundary conditions:

$$a_1y(x_1) + a_2y(x_2) + a_3y'(x_1) + a_4y'(x_2) = A$$
$$b_1y(x_1) + b_2y(x_2) + b_3y'(x_1) + b_4y'(x_2) = B,$$

in connection with certain ordinary linear differential equations of the second order which contain a parameter. Some of the results of the paper were published in the *Comptes rendus*, April 27, 1905.

13. Professor Morley's paper is in abstract as follows: The fixed points of a collineation belonging to a pencil, in a plane, are on a cubic curve ϕ; and the fixed lines are on a cubic curve f. Thus ϕ and f admit an infinity of Poncelet triangles. By joining every point to the corresponding point in a collineation we pick out of the ∞^3 elements of the plane a nexus of ∞^2 elements. The elements common to two nexuses form the two cubics. Each triangle forms with every element a constant double ratio. The two cubics have 6 contacts, 6 simple common points, and 6 simple common lines.

14. In Dr. Sisam's paper the equation of a plane curve in one to one correspondence with the nodal curve of a rational scroll was obtained. Let the equations of the scroll be: $x_i = a_i(u) + vb_i(u)$ $(i = 1, 2, 3, 4)$. If the generators determined by $u = u$ and $u = u'$ intersect, then the determinant $|\, a_1(u)\, a_2(u')\, b_3(u)\, b_4(u')\,| = 0$. This equation, after division by $(u - u')^2$, may be expressed in terms of $u + u'$ and uu' only. Call it $F(u + u', uu') = 0$. Put $u + u' = \xi$, $uu' = \eta$ and consider $F(\xi, \eta) = 0$ as the equation of a plane curve. It then follows that each irreducible component of $F(\xi, \eta) = 0$ determines an irreducible component of the nodal curve on the surface. If the latter is double only, then the number of times it is cut by an arbitrary generator equals the order, and its genus equals the genus of the component of $F = 0$ which determines it.

The components of the double developable and their properties analogous to those of the double curve are also determined by the components of $F(\xi, \eta) = 0$. The equation in u found by putting $u' = u$ in $F(u + u', uu') = 0$ determines the parameters of the torsal generators.

15. If the radial and axial displacements u and w caused by the rotation of a circular disc of thickness $2h$ and radius R, at a point (v, z), be expressed as follows:

$$(1) \qquad u = -\frac{\rho\omega^3}{8(2\mu + \lambda)} r^3 + \sum_0^\infty u_n z^{2n},$$

$$(2) \qquad w = \quad w_n z^{2n+1};$$

if, moreover, we put

$$(3) \qquad z_n = h^{2n}\left\{ (2\mu + \lambda)\frac{1}{r}\frac{dru_n}{dr} + \lambda(2n+1)w_n \right\},$$

$$(4) \qquad y^n = h^{2n}\left\{ \lambda\frac{1}{r}\frac{dru_n}{dr} + (2\mu + \lambda)(2n+1)w_n \right\},$$

then we have, to determine x_n and y_n, the system of differential equations

$$\begin{aligned} &(5)\ h^2 Dx_n = (2n+1)(2n+2)y_{n+1}, \\ &(6)\ h^2 Dy_n = -(2n+1)(2n+2)(x_{n+1} + 2y_{n+1}) \end{aligned} \quad (n = 0, 1, 2, \cdots),$$

where D indicates the operation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}.$$

The surface conditions for $z = \pm h$ furnish two equations, viz.:

$$(7) \qquad \sum \frac{x_n}{2n+1} = C, \qquad \sum y_2 = \frac{\lambda\rho\omega^2}{2(2\mu+\lambda)}r^2.$$

The integration of the system (5)–(6), together with equations (7), gives the following results:

$$(8) \qquad \begin{gathered} (n > 1)\ x_n = \sum C_{n,e} J_0\left(\frac{\lambda_e}{h}r\right), \qquad (n \geqq 1)\ y_n = \sum C'_{n,e} J_0\left(\frac{\lambda_e}{h}r\right) \\ x_0 = C + \sum C_{0,e} J_0\left(\frac{\lambda_e}{h}r\right), \\ x_1 = -\frac{\lambda\rho\omega^2}{2\mu+\lambda}h^2 + \sum C_{1,e} J_0\left(\frac{\lambda_e}{h}r\right), \\ y_0 = \frac{\lambda\rho\omega^2}{2(2\mu+\lambda)}\mu^2 + \sum C'_{0,e} J_0\left(\frac{\lambda_e}{h}r\right), \end{gathered}$$

where the λ_e are the roots of the transcendental equation

$$(9) \qquad \sin 2ix + 2ix = 0$$

and the coefficients $C_{n,e}$, $C'_{n,e}$ are given by the formulas

$$C_{n,e} = \left(1 + \frac{2n}{1 + \cos^2 i\lambda_e}\right) \frac{\lambda_e^{2n}}{2n\,!} C_{0,e},$$

$$C'_{n,e} = \left(\frac{\sin^2 i\lambda_e - 2n}{1 + \cos^2 i\lambda_e}\right) \frac{\lambda_e^{2n}}{2n\,!} C_{0,e},$$

while the coefficients C and the C_{0e} are determined from the two remaining surface conditions for $r = R$.

Once the x_n and y_n are determined, the w_n are obtained from (3)–(4) directly and the u_n by a simple integration. Thus will be found the displacements u and w and by their means the strains and stresses at any point of the disc.

This solution of the problem does not readily lend itself to practical applications on account of its intricate form. A simplified solution will be the subject of a subsequent paper.

16. The group of all quaternary linear homogeneous substitutions of determinant unity modulo p is of order $p^6(p^4 - 1)(p^3 - 1)(p^2 - 1)$. It is shown by Professor Dickson that any subgroup of order a multiple of p^6 is contained in a group conjugate with one of three maximal subgroups

$$\begin{array}{rl} \text{(I)} & (a_{ij}),\ a_{13} = a_{14} = a_{23} = a_{24} = 0,\ |a| = 1\,; \\ \text{(II)} & (a_{ij}),\ a_{14} = a_{24} = a_{34} = 0,\ |a| = 1\,; \\ \text{(III)} & (a_{ij}),\ a_{12} = a_{13} = a_{14} = 0,\ |a| = 1. \end{array}$$

Here II and III are simply isomorphic groups of order $p^3(p^3 - 1)(p^3 - p)(p^3 - p^2)$, while I is of order $p^6(p^2 - 1)^2(p - 1)$. Next, let H be a subgroup of order $p^5 N$, N prime to p, $p > 2$. If its G_{p^5} is invariant, H is contained in (a_{ij}), $a_{ij} = 0 (j > i)$ except a certain $a_{s-1,s}$. If G_{p^5} is not invariant, H is conjugate with one of the types: (I) with the $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ defining a binary group of order prime to p; (I) with the same restriction on the $\left(\begin{smallmatrix} a_{33} & a_{34} \\ a_{34} & a_{44} \end{smallmatrix}\right)$; (I) with

$$a_{11} = \pm e^{-1} a_{33},\ a_{12} = \pm \mu^{-1} e^{-1} a_{34},\ a_{21} = \pm \mu e^{-1} a_{43},\ a_{22} = \pm e^{-1} a_{44},$$

where $e = a_{33}a_{44} - a_{34}a_{43}$, and μ is a fixed integer, either 1 or a particular quadratic non-residue of p. According to the restrictions placed on the sign $\pm$ and on e, the last group is of order $p^5(p^2-1)t$, t any chosen factor of $2(p-1)$. The paper has been offered to the *American Journal of Mathematics.*

17. Of the algebras in which all but one of the postulates of a field hold (*Transactions*, 1905, page 201), the most interesting are those in which the commutativity or associativity of multiplication is not assumed. In his second paper, Professor Dickson treats at length finite algebras whose elements form a group under addition, whose elements $\neq 0$ form a group under multiplication, and for which the left-hand distributive law holds. (If also the right-hand distributive law holds, the algebra is a field.) It is shown that the elements may be given the notation $(a_1, \cdots, a_n)$, each a_i an integer taken modulo p a prime, and that $(a_i) + (b_i) = (a_i + b_i)$. The algebra is a field if $p^n - 1$ and n are relatively prime, or if $n = 1$. In the remaining cases, non-field algebras exist; a typical example is

$$(a, b) \times (x, y) = (ax + evby,\ bx + eay),$$

where v is a fixed quadratic non-residue of p, and $e = \pm 1$ according as $a^2 = vb^2$ is a quadratic residue or non-residue of p.

In the second type of algebra, the elements form a group under addition, the product of any two is in the set, the right-hand identity and inverse occur, and multiplication is commutative and distributive; the omitted postulate is associativity of multiplication. If the algebra is finite the elements are $(a_1, \cdots, a_n)$, the a_i taken modulo p, prime, and $(a_i) + (b_i) = (a_i + b_i)$. If $n < 3$, the algebra is a field. Non-field algebras are found for n any multiple of 3. The remaining cases were not examined. If F is any finite or infinite field not having modulus 2, an algebra having all the required properties is that with the units 1, i, j, and coördinates in F, where $i^2 = j$, $ij = ji = b + \beta i$, $j^2 = -\beta^2 = 8bi - 2\beta j$, where b and β are such that $x^3 = b + \beta x$ is irreducible in F. In this linear, non-associative algebra, every element has an unique inverse. When F is the field of integers modulo p, this algebra is the only such non-field algebra for $p = 3, 5, 7$ or 11, the cases examined exhaustively.

18. The curves discussed by Professor Snyder are rational, of order n, and have two tangents of $n - 1$ pointic contact at

A and B. The osculating plane at any point P cuts the curve in $n-3$ points Q_i. The surface generated by the lines PQ_i is composite, each factor being rational and of order $2n-4$. The given curve is tacnodal on each such surface, and the residual curve breaks up into $n-2$ factors, each being cut twice by every generator. If n is even, one of the points Q_k is the harmonic conjugate of P as to A and B. The lines PQ_k generate a scroll having AB for multiple directrix, upon which the given curve is an asymptotic line.

20. In the geometry of birational transformations in a plane or other rational surface, the Cremona group is a subgroup of the general algebraic or Riemann group, hence the former may have some invariants distinct from those of the latter. Professor White points out the fact that this is not so when the object to which the transformations are applied is a single rational curve on the surface.

21. Professor Moulton's paper discusses the existence, properties, and construction of certain periodic solutions of the problem of three bodies in which the orbits reduce to circles with the vanishing of certain parameters. With proper specialization of the parameters the problem is that of the periodic orbits of two mutually disturbing finite planets; with another specialization, the problem of the periodic solution of a finite satellite disturbed by the sun; and with certain further limitations, the moon's variational orbit. The results are developed literally so as to apply to any case in the various types, and numerical illustrations are exhibited. They include the moon's variational and parallactic orbit carried out to an order of accuracy which leaves nothing to be desired, and two classes of the periodic orbits discovered by Darwin (*Acta Mathematica,* volume 21) from numerical experiments.

22. Dr. Haskins's paper indicates the simplifications which the use of Ricci's methods of covariant differentiation permit to be made in Forsyth's recent work on differential invariants of quadratic differential forms.

23. The main part of Professor Huntington's paper gives a systematic elementary account of the purely ordinal theory of the continuum, as begun by Dedekind in 1872, and completed by Georg Cantor in 1895. An appendix on Cantor's "well-

ordered" classes and the transfinite numbers will serve as an introduction to the study of these most recent accessions to the list of mathematical concepts. The matter is of interest to the philosopher as well as to the mathematician; and the present exposition is intended especially for the general student of scientific method, who, without technical mathematical training, wishes to keep in touch with the modern development in the logic of mathematics. The mathematical prerequisites have been reduced to a minimum; the demonstrations are given in full; all new concepts are defined explicitly by sets of independent postulates; and in connection with each definition numerous examples are given, to illustrate, in a concrete way, not only the systems which have, but also those which have not, the property in question. The paper is being published in the *Annals of Mathematics* for July and October, 1905.*

The chapter headings are as follows: On classes in general; Ordered classes, or "series"; Discrete series, especially the type of order exhibited by the natural numbers; Dense series, especially the type of the rational numbers; Continuous series, especially the type of the real numbers; Continuous series in more than one dimension, with a note on multiply-ordered classes. An appendix treats of Cantor's "well-ordered" classes, and the transfinite numbers, and there is an index of technical terms. The paper contains also a bibliography of Cantor's writings on these subjects.

F. N. COLE,
Secretary.

A SET OF GENERATORS FOR TERNARY LINEAR GROUPS.

BY MISS IDA MAY SCHOTTENFELS.

(Read before the American Mathematical Society, September 17, 1904.)

The following is a proof that the substitutions

$$c_k:\quad x'_i = x_{i+1},\ x'_k = x_1 + hx_2 \qquad (i = 1, 2, \cdots k-1;\ h = 0, 1)$$

generate (1) the ternary linear substitution group with integral coefficients of determinant unity, and (2) the linear and linear-fractional Galois field groups $[2^1]$.*

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* Reprints of this and other papers published in the *Annals* can be ordered from the Publication Office of Harvard University.

In recent literature on linear and linear-fractional congruence groups, entrance to such groups has frequently been made by means of the above substitutions.†

In the proof of (1) the methods of Burnside ‡ are employed, and in (2) an analogue of this method is made use of for the Galois field $[2^1]$.

1. All substitutions of the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3, \end{aligned}$$

with coefficients rational integers of determinant unity can be formed by combinations of the two following substitutions:

$$T = (x_2 \cdot x_3 \cdot x_1), \qquad S = (x_2 \cdot x_3 \cdot x_2 + x_1).$$

The following combinations only are made use of in this proof:

$$\begin{gathered} P = TST = (x_1 \cdot x_2 + x_3 \cdot x_3), \quad Q = T^2S = (x_1 \cdot x_2 \cdot x_3 + x_1), \\ U = PV^{-1}P = (x_1 \cdot x_3 \cdot - x_2),\ W = ST^{-1} = (x_1 + x_2 \cdot x_2 \cdot x_3), \\ V = (T^2S^2T^{-1})^2(ST^{-1})^{-2}(T^2S)^{-2} = (x_1 \cdot x_2 \cdot x_3 + x_2). \end{gathered}$$

From the form of P, Q, W, V and $T^n VT^{2n}$ $(n = 0, 1, 2)$ it is evident that T and S give rise to six substitutions in which two of the symbols are unchanged and the third is reinforced by either of the unchanged symbols.

From the form $T^n U^m T^{2n}$ $(n = 0, 1, 2; m = 1, 2, 3)$, derived from U, it is evident that T and S give rise to all substitutions in which one of the symbols remains unchanged.

The remainder of this proof consists in showing that corresponding to any substitution whatever, say Σ, a substitution

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[*] Krazer, *Annali di Matematica*, (2) vol. 12, pp. 283–300. Kronecker, *Monatsberichte*, Berlin Akademie, Oct. 15, 1866, p. 597.

[†] Moore, "Concerning the general equations of the seventh and eighth degrees." *Mathematische Annalen*, vol. 51, p. 436. Schottenfels, *Annals of Mathematics*, 2d ser., vol. 1, No. 3; BULLETIN, 2d ser., vol. 6, pp. 440–443.

[‡] Burnside, *Messenger of Mathematics*, vol. 24, p. 109.

can be formed by a finite number of combinations and repetitions of T and S which, performed after Σ, leads to identity, and which is therefore Σ^{-1}. The inverse of Σ^{-1}, which in turn can be formed by a finite number of combinations and repetitions of T and S, is therefore equal to Σ.

Let

$$\Sigma = (\alpha x_1 + \beta x_2 + \gamma x_3,\ \alpha' x_1 + \beta' x_2 + \gamma' x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3)$$

where $\gamma \geqq \gamma'$.

$$\Sigma W^m = ([\alpha + m\alpha']x_1 + [\beta + m\beta']x_2 + [\gamma + m\gamma']x_3,\\ \alpha' x_1 + \beta' x_2 + \gamma' x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

or

$$\Sigma W^m = (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3,\ \alpha' x_1 + \beta' x_2 + \gamma' x_3,\\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

where the integer m, positive or negative, can always be chosen so that γ_1 is numerically less than γ'. Hence we have

$$\Sigma W^m (T^2 V T)^{m'} = (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3,\ \alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3,\\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3),$$

where m' can be chosen so that γ_2 is numerically less than γ_1.

After a finite number of such substitutions, a substitution must be arrived at in which one of the two coefficients corresponding to γ and γ' in Σ is zero; and, if it is the coefficient corresponding to γ', combining with T^2UT gives a substitution in which the coefficient corresponding to γ vanishes and Σ is reduced to

$$\Sigma' = (ax_1 + bx_2,\ a'x_1 + b'x_2 + c'x_3,\ \alpha'' x_1 + \beta'' x_2 + \gamma'' x_3).$$

A precisely similar series of substitutions will reduce this to a form in which the coefficient corresponding to c' vanishes; and on the other hand since the determinant of the substitution remains unity, the coefficient corresponding to γ'' must become ± 1. By combining with U^2 if necessary, this last coefficient may be made $+1$, and since the coefficients a and b are unaffected by this set of substitutions, Σ' becomes

$$\Sigma'' = (ax_1 + bx_2,\ a_1 x_1 + b_1 x_2,\ a_2 x_1 + b_2 x_2 + x_3).$$

By repeating a set of substitutions similar to the first set, Σ'' may be reduced to a substitution in which b vanishes, and b_1 becomes $+1$; and since the determinant remains unity, a at the same time becomes $+1$, and Σ'' becomes

$$\Sigma''' = (x_1,\ a_1' x_1 + x_2,\ a_2 x_1 + b_2 x_2 + x_3).$$

Finally $(T^2 UT)^{-a_1'} Q^{-a_2} . V^{-b_2}$ reduces Σ''' to

$$\Sigma^{\mathrm{IV}} = (x_1,\ x_2,\ x_3) \text{ or identity.}$$

2. All matrices of the form (a_{ij}) $(ij = 1, 2, 3)$

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

where a_{ij} are marks of the Galois Field $[2^1]$ and where the matrix (a_{ij}) is such that its determinant is unity can be formed by combinations of the two following matrices

$$T = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad S = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

The following combinations only are made use of in this proof:

$$P = TST = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \qquad Q = T^2 S = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \qquad W = ST^{-1} = \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$V = (T^2 S^2 T^{-1})^2 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \qquad U = PV^{-1}P = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

As in case (1), P, Q, W, V and the form $T^n V T^{2n}$ $(n = 0, 1, 2)$ give rise to six substitutions in which two of the symbols are unchanged and the third is reinforced by either of the unchanged symbols. Also the form $T^n U T^{2n}$ $(n = 0, 1, 2)$ contains all the substitutions in which one of the symbols remains unchanged.

The remainder of this proof proceeds as in case (1). Let

$$\Sigma = \begin{array}{ccc} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{array};$$

if $\gamma < \gamma'$, then $\gamma = 0$ and Σ becomes

$$\Sigma' = \begin{array}{ccc} \alpha & \beta & 0 \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{array};$$

otherwise $\gamma \geqq \gamma'$.

$$\Sigma W = \begin{array}{ccc} \alpha + \alpha' & \beta + \beta' & \gamma + \gamma' \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{array} \quad \text{or} \quad \Sigma W = \begin{array}{ccc} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{array}.$$

(1) If $\gamma = \gamma'$, then $\gamma_1 = 0$. (2) If $\gamma > \gamma'$, then $\gamma' = 0$.

In (1) ΣW reduces to Σ'. In (2) we combine ΣW with $T^2 UT$ and again get

$$\Sigma' = \Sigma W T^2 U T = \begin{array}{ccc} \alpha' & \beta' & \gamma' \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha'' & \beta'' & \gamma'' \end{array},$$

where $\gamma' = 0$,

$$\Sigma' P = \begin{array}{ccc} \alpha' & \beta' & 0 \\ \alpha_1 + \alpha'' & \beta_1 + \beta'' & \gamma_1 + \gamma'' \\ \alpha'' & \beta'' & \gamma'' \end{array} \quad \text{or} \quad \Sigma' P = \begin{array}{ccc} \alpha' & \beta' & 0 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha'' & \beta'' & \gamma'' \end{array}.$$

If $\gamma_1 < \gamma''$, then $\gamma_1 = 0$, and Σ' becomes

$$\Sigma'' = \begin{array}{ccc} \alpha' & \beta' & 0 \\ \alpha_1 & \beta_1 & 0 \\ \alpha'' & \beta'' & \gamma'' \end{array}.$$

In the remaining case $\gamma_1 \geqq \gamma''$.
(1) If $\gamma_1 = \gamma''$ then $\gamma_2 = 0$. (2) If $\gamma_1 > \gamma''$ then $\gamma'' = 0$.

In (1) $\Sigma' P$ reduces to Σ''. In (2) $\Sigma' P$ combined with U gives

$$\Sigma'' = \begin{matrix} \alpha' & \beta' & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

$$\Sigma'' W = \begin{matrix} \alpha' + \alpha'' & \beta' + \beta'' & 0 \\ \alpha'' & \beta'' & 0, \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix} \quad \text{or} \quad \Sigma'' W = \begin{matrix} \alpha_3 & \beta_3 & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2' \end{matrix}$$

If $\beta' < \beta''$ then $\beta' = 0$ and Σ'' becomes

$$\Sigma''' = \begin{matrix} \alpha' & 0 & 0 \\ \alpha'' & \beta'' & 0. \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

In the remaining case $\beta' \geqq \beta''$.

(1) If $\beta' = \beta''$ then $\beta_3 = 0$. (2) If $\beta' > \beta''$ then $\beta'' = 0$.

In (1) $\Sigma'' W$ reduces to Σ'''. In (2) $\Sigma'' W$ combined with $T^2 UT$ gives

$$\Sigma''' = \begin{matrix} \alpha'' & 0 & 0 \\ \alpha_3 & \beta_3 & 0, \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix}$$

Finally $\Sigma''' T^2 UTQV$ gives Σ^{IV}, which is identity,

$$\Sigma^{\text{IV}} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0. \\ 0 & 0 & 1 \end{matrix}$$

In conclusion, the generators T and S give rise to the simple group of order 168 with generational definition as follows:

$$\text{G}_{168} = \{ T^3 = S^7 = (ST^2)^2 = (TS^2)^4 = \text{I} \}.^*$$

NEW YORK, *June*, 1905.

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[*] Schottenfels: "Upon the non-isomorphism of two simple groups of order 8!/2 or 20160," BULLETIN, 2d ser. vol. 8, no. 1, p. 26, § 2.

NOTE ON THE STRUCTURE OF HYPERCOMPLEX NUMBER SYSTEMS.

BY DR. SAUL EPSTEEN.

(Read before the American Mathematical Society, September 7, 1905.)

In the *Transactions* for April, 1905, on page 176 * the following theorem (No. III.) was enunciated:

Let $E, E_1, E_2, \cdots$ be a normal series of subalgebras of E (i. e., E_r is a maximal invariant subalgebra of E_{r-1}, $E_0 = E$) and let $K_1, K_2, \cdots$ be a series of complementary algebras, such that K_r accompanies E_{r-1} and is complementary to E_r. Under these assumptions the series $K_1\ K_2, \cdots$ is, apart from the order, independent of the choice of the series $E, E_1, E_2, \cdots$. In other words, if $E, E'_1, E'_2, \cdots$ is any other normal series, the complementary series of algebras $K'_1, K'_2, \cdots$ which it defines is the same as the series $K_1, K_2, \cdots$, apart from the sequence.

With the exception of this theorem and the corresponding one where a *chief series* is used in place of the *normal series* all the proofs were made in a symbolic notation which is a generalization of one introduced by Frobenius in the theory of abstract groups. † In demonstrating the above proposition we made use of the *characteristic constants* $\gamma_{i_1 i_2 i_3}$ of the system

$$E \equiv e_1 \cdots e_n \quad : \quad (e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3}),$$

which, owing to the associative law, satisfy the n^3 conditions

$$\sum_{i_3} (\gamma_{i_1 i_2 i_3} \gamma_{i_3 i_4 i_5} - \gamma_{i_1 i_3 i_5} \gamma_{i_2 i_4 i_3}) = 0 \quad (i = 1, \cdots, n).$$

In the present note it is shown how this theorem can also be demonstrated in the symbolic notation without recourse to the γ's.

In order to avoid repetitions I employ the same numbering of theorems and equations, and also the same letters and symbols as in the above quoted paper, which this note supplements.

According to the demonstration of the theorem I, we have (if $E_1 \mp E'_1$)

$$E = E_1 + E'_1.$$

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* Epsteen-Wedderburn, "On the structure of hypercomplex number systems," *Transactions Amer. Math. Society*, vol. 6, pp. 172–178.

† Frobenius, *Berliner Sitzungsberichte*, 1895, p. 164.

Let

$$E_1 \frown E_1' = F_1;$$

then

$$E_1 = F_1 + D_1 \qquad (F_1 \frown D_1 = 0),$$

$$E_1' = F_1 + D_1' \qquad (F_1 \frown D_1' = 0),$$

and it is evident that

$$E = F_1 + D_1 + D_1'.$$

By theorem 2, F_1 is a maximal invariant subalgebra of E_1 and E_1'.

If $F_1, F_2, F_3, \cdots$ is a normal series of F_1, then $E, E_1, F_1, F_2, \cdots$ and $E, E_1', F_1, F_2, \cdots$ are two new normal series of E. For this purpose it is merely necessary to prove that the complementary algebras defined by E, E_1, F_1 and E, E_1', F_1 are the same.

The complementary algebra of $E_1 (= F_1 + D_1)$ with respect to E is

$$(9') \quad \begin{aligned} E(\operatorname{mod} E_1) &= F_1 + D_1 + D_1' (\operatorname{mod} F_1 + D_1) \\ &= D' (\operatorname{mod} F_1 + D_1) = D_1' (\operatorname{mod} F_1), \end{aligned}$$

since

$$(F_1 + D_1)^2 \leqq F_1 + D_1 \text{ and } D_1'^2 \frown D_1 = 0.$$

Furthermore, the complementary algebra of F_1 with respect to E_1 is

$$(10') \qquad E_1 (\operatorname{mod} F_1) = F_1 + D_1 (\operatorname{mod} F_1) = D_1 (\operatorname{mod} F_1).$$

Now it is seen in the same manner that the complementary algebra of E_1' with respect to E is

$$(10') \qquad D_1 (\operatorname{mod} F_1)$$

and that the complementary algebra of F_1 with respect to E_1' is

$$(9') \qquad D_1' (\operatorname{mod} F_1)$$

and, therefore, the normal complementary series defined by E, E_1, F_1 is identical, apart from the sequence, with the normal complementary series defined by E, E_1', F_1.

It now becomes necessary to show that the normal series E, $E_1, E_2, \cdots$; $E, E_1, F_1, F_2, \cdots$ define the same complementary series, and similarly for $E, E'_1, E'_2, \cdots$; $E, E'_1, F_1, F_2, \cdots$. This amounts to proving the theorem for the algebras E_1 and E'_1, which involves a finite number of repetitions of the above proof.

A chief or principal series $E, P_1, P_2, \cdots$ of an algebra being defined as one in which P_r is the maximal subalgebra of P_{r-1} which is invariant is $E\,(P_0 = E)$ it is easily shown in the symbolic notation, by exactly the same process as for the normal series, *that the system of complementary algebras* $C_1, C_2, \cdots$ *is independent of the chief series selected.*

In the case of the normal series the complementary algebras $K_1, K_2, \cdots$ are necessarily simple, but this is not true of the complementary algebras $C_1, C_2, \cdots$ in the case of the chief series.

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A GEOMETRIC PROPERTY OF THE TRAJECTORIES OF DYNAMICS.

BY DR. EDWARD KASNER.

(Read before the American Mathematical Society, September 7, 1905.)

Suppose that the force acting on a particle whose coordinates are x, y, z, produces an acceleration having the components $\phi\,(x, y, z)$, $\psi\,(x, y, z)$, $\chi\,(x, y, z)$. The equations of motion are then

$$(1)\qquad \ddot{x}=\phi\,(x, y, z),\ \ddot{y}=\psi\,(x, y, z),\ \ddot{z}=\chi\,(x, y, z),$$

where dots denote differentiation with respect to the time t. In such a field of force the initial position and initial velocity completely determine the trajectory. The totality of trajectories thus constitutes a quintuply* infinite system of space curves.

Consider now those trajectories obtained by starting the particle at a given point P in a given direction. These form a simply infinite system since the initial speed v is arbitrary.

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* The only exception arises in the trivial case where the force is everywhere zero. Then the trajectories are the fourfold infinity of straight lines.

The osculating plane at P must contain both the given direction and the direction of the force acting at P; hence the ∞^1 trajectories have the same osculating plane. The osculating sphere, however, varies from trajectory to trajectory. We now prove that, no matter what the law of force, the locus of centers of these ∞^1 osculating spheres is a straight line.

The formulas for the center of the osculating sphere of any space curve are[†]

$$\begin{aligned} X &= x + r^2x'' - \rho r r'(y'z'' - z'y''), \\ Y &= y + r^2y'' - \rho r r'(z'x'' - x'z''), \qquad (2) \\ Z &= z + r^2z'' - \rho r r'(x'y'' - y'x''), \end{aligned}$$

where accents denote differentiation with respect to the arc s; r is the radius of curvature, and ρ the radius of torsion. These are determined by the familiar formulas

$$1/r^2 = x''^2 + y''^2 + z''^2 = \Sigma x''^2, \tag{3}$$

$$\frac{1}{\rho} = -\frac{1}{\Sigma x''^2}\begin{vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ z' & z'' & z''' \end{vmatrix}. \tag{4}$$

Differentiating (3), we find

$$r' = -r^3\Sigma x''x'''. \tag{5}$$

To apply these formulas to the trajectories of dynamics, it is necessary to replace arc derivatives by time derivatives. The transformation formulas are

$$\begin{aligned} x' &= \dot{x}/v, \\ x'' &= (v\ddot{x} - \dot{v}\dot{x})/v^3, \qquad (6) \\ x''' &= [v(v\dddot{x} - \ddot{v}\dot{x}) - 3\dot{v}(v\ddot{x} - \dot{v}\dot{x})]/v^5, \end{aligned}$$

with similar results for y and z.

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[*] See Scheffers, Theorie der Curven, Leipzig (1901), p. 234.

Here v denotes the velocity, so that

$$(7) \qquad v^2 = \Sigma \dot{x}^2, \quad v\dot{v} = \Sigma \dot{x}\ddot{x}, \quad v\ddot{v} + \dot{v}^2 = \Sigma \dot{x}\dddot{x} + \Sigma \ddot{x}^2.$$

By means of formulas (2) to (7), the coördinates of the center of the osculating sphere may be expressed in terms of x, y, z and their time derivatives of first, second and third orders.

We now simplify the discussion by means of a particular choice of axes. Let the given point P be taken as origin; the given direction, as axis of x; and let the fixed osculating plane be the plane xy. Since the initial velocity is directed along the axis of x, we have

$$\dot{x}, \dot{y}, \dot{z}, = v, 0, 0.$$

The force acts in the plane xy, hence the acceleration components are

$$\ddot{x}, \ddot{y}, \ddot{z} = \phi, \psi, 0,$$

where ϕ, ψ are constants since $x, y, z, = 0, 0, 0$. The third derivatives are

$$\dddot{x}, \dddot{y}, \dddot{z}, = v\phi_x, v\psi_x, v\chi_x.$$

From (6), we can now find the arc derivatives

$$(8) \qquad \begin{aligned} x', y', z' &= 1, 0, 0, \\ x'', y'', z'' &= 0, \quad \psi/v^2, 0, \\ x''', y''', z''' &= (-\psi^2, v^2\psi_x - 3\phi\psi, v^2\chi_x)/v^4. \end{aligned}$$

Substituting these values in (3), (4), (5), we have

$$(9) \qquad r = \frac{v^2}{\psi}, \quad \rho = -\frac{\psi}{\chi_x}, \quad r' = \frac{3\phi\psi - v^2\psi_x}{\psi^2}.$$

Finally, from (2), we find

$$(10) \qquad X = 0, \ Y = \frac{v^3}{x}, \ Z = \frac{3\phi\psi - v^2\psi_x}{\psi\chi}$$

Here we note that X, Y, Z are linear functions of the parameter v^2. Elimination thus gives the straight line

$$(11) \qquad X = 0, \ Z = \frac{3\phi}{\chi_x} - \frac{\psi_x}{\chi_x} Y$$

as the required locus of centers.

Incidentally, the value of ρ obtained in (9) shows that the torsion, unlike the curvature, is independent of v.

An arbitrary field of force (1) *produces* ∞^5 *trajectories, of which* ∞^1 *pass through a given point in a given direction. These* ∞^1 *trajectories have, at the given point, a common osculating plane and a common torsion. The locus of centers of their osculating spheres is a straight line. Thus every field of force gives rise to a correspondence between the direction elements and the straight lines of space.*

COLUMBIA UNIVERSITY,
August, 1905.

ON THE POSSIBLE NUMBERS OF OPERATORS OF ORDER 2 IN A GROUP OF ORDER 2^m.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 7, 1905.)

It is well known that every group of order 2^m which contains only one operator of order 2 is either cyclic or it is composed of the cyclic group of order 2^{m-1} and 2^{m-1} operators of order 4 transforming each operator of this cyclic group into its inverse.[*] There are exactly two such groups for every value of $m > 2$. When $m = 3$ the latter of these two is the quaternion group, and when $m < 3$ the cyclic group is the only one that contains only one operator of order 2.

The groups of order 2^m in which the number of all the operators of order 2 is $\equiv 1$ mod. 4 have been determined incidentally in a recent paper.[†] Such groups exist only when the number of operators of order 2 is of the form $2^k + 1$, and there are exactly two possible groups for every arbitrary value of k. One of these is the dihedral rotation group of order 2^{k+1}, and the other is obtained by adding to the cyclic group of order 2^{k+1} an operator of order two which transforms each of its operators into its $(2^k - 1)$th power. Just half of the additional operators are of order two and the others are of order 4.

For instance, there are just two groups whose orders are of the form 2^m and which contain just five operators of order two;

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[*] Burnside, Theory of groups, 1897, p. 75.

[†] *Transactions Amer. Math. Society*, vol. 6 (1905), p. 62.

one of these is the well known octic group, or the group of movements of the square, while the other is of order 16 and includes the cyclic group of order 8 in addition to four operators of each of the orders 2 and 4 transforming each operator of this cyclic subgroup into its third power. The latter constitute two complete sets of conjugates, while the operators whose orders exceed two in the cyclic subgroup of order 8 are conjugate in pairs. In general, the operators whose orders exceed two in the cyclic subgroup of orders 2^k or 2^{k+1} respectively are conjugate in pairs while the remaining operators constitute two complete sets of conjugates.

It is well known that every group whose order is divisible by 2 contains an odd number of operators of order 2. Moreover, it is clearly possible to construct a group which contains any arbitrary odd number of operators of order 2. To construct a group which contains exactly $2n+1$ operators of this order it is only necessary to extend the cyclic group of order $2n+1$ by an operator of order 2 which transforms each operator of this cyclic group into its inverse. From the theorem mentioned in the second paragraph it follows that it is impossible to construct a group whose order is of the form 2^m and which contains an arbitrary odd number of operators of order 2.

In view of the popular saying that 13 is an unlucky number, it is of interest to note that *there is no group whose order is of the form 2^m and which contains exactly 13 operators of order 2*, but there are groups which contain exactly any other odd number, less than twenty-one, of such operators. The truth of this statement follows almost directly from the second paragraph. It will be included among the results of the developments which follow. In these developments it will be assumed that the orders of all the groups under consideration are powers of 2.

The dihedral rotation group of order 2^{k+1} contains exactly 2^k+1 operators of order 2, whenever $k > 0$. By forming the direct product of this dihedral rotation group and the abelian group of order 2^β and of type $(1, 1, 1, \cdots)$ there results a group in which the number of operators of order 2 is exactly $2^{k+\beta}+2^{\beta+1}-1$. As the values of β and k are arbitrary except that $k \neq 0$ it follows that *it is possible to construct groups in which the number of operators of order 2 is exactly 2^x+2^y-1, where x and y can have any arbitrary pair of integral values with the exception of* 0, 1. When the groups are abelian $x=y$, but the converse is not necessarily true.

From the preceding paragraph it follows that it is possible to construct a group of order 2^m in which the number of operators of order 2 is the sum of two arbitrary powers of 2 diminished by one. We shall now prove that it is possible to construct a group in which the number of these operators is the sum of three arbitrary powers of two diminished by one, provided the smallest of these powers is divisible by 4. It is clear that we may restrict ourselves to the case when the three powers are distinct since the other cases reduce to those considered in the preceding paragraph.

To prove this theorem it is convenient to consider the group (G_1) of order 2^{a+2} which is obtained by taking all the operators of the holomorph of the cyclic group of order 2^a which transform this cyclic group into its $1, 2^{a-1}-1, 2^{a-1}+1$, and 2^a-1 powers respectively. These powers are distinct when $a > 2$. In this case there are $1, 2^{a-1}, 2$ and 2^a operators of order 2 which transform the cyclic group into the respective powers mentioned above. It is clearly possible to establish a $(2^a, 2^\beta)$ isomorphism between G_1 and the dihedral rotation group of order $2^{\beta+2}$ in such a manner as to obtain a group (G) which contains $3 + 2^{\beta+1} + 2^{a+\beta}$ operators of order 2, whenever $\beta > 0$.

As a and β are arbitrary independent positive integers, with the restriction that $a > 2$, it follows that the number of operators of order 2 in G is the sum of two powers of 2, whose indices differ by at least two units but are otherwise arbitrary, increased by 3. In G_1 the number of these operators is the sum of two powers of 2, whose indices differ by one but are otherwise arbitrary, increased by 3. Hence by forming the direct product of an abelian group of order 2^γ and of type $(1, 1, 1, \cdots)$ into either G or G_1 we obtain *a group in which the number of operators of order 2 is $2^x + 2^y + 2^z - 1$, where x, y, z satisfy the condition $x > y > z > 1$ are but otherwise arbitrary.* This proves the theorem in question.

It has recently been proved that the number of operators of order 2 in a group of order 2^m, in which more than half the operators are of order 2, is always the sum of two powers of 2 diminished by one, and that it is possible to construct such a group in which the number of operators of order 2 is an arbitrary number of this form. Hence it follows from the preceding paragraph that groups in which less than half the operators are of order 2 may involve a number of operators of this order which could not be the exact number of these operators in any group in which more than half the operators are of order 2.

A $(2^{\alpha}, 2^{\beta})$ isomorphism between G_1 and the direct product of the dihedral rotation group of order $2^{\beta+1}$ into an operator of order 2 can be established in such a manner as to obtain a group in which the number of operators of order 2 is either $3 + 2^{\alpha} + 2^{\beta+1} + 2^{\alpha+\beta}$, or $3 + 2^{\alpha+1} + 2^{\beta+1} + 2^{\alpha+\beta-1}, \beta > 0$. In fact, it is possible to form other such isomorphisms, but these two seem especially useful in this connection. Moreover, by establishing a $(2^{\alpha}, 2^{\beta})$ isomorphism between G_1 and a group of order $2^{\beta+2}$ which is constructed in the same way as G_1, we arrive at groups which contain any of the following three numbers of operators of order 2: $3 + 2^{\alpha} + 2^{\beta} + 2^{\alpha+\beta}$, $3 + 2^{\alpha+1} + 2^{\beta+1} + 2^{\alpha+\beta-2}$, $3 + 2^{\alpha+1} + 2^{\beta} + 2^{\alpha+\beta-1}$.

From the above results it follows directly that there are groups of order 2^m which contain any prescribed number of operators of order 2 which satisfies the conditions that it is $\equiv 3$ mod. 4 and less than 124. By other considerations this limit can readily be extended, but my methods seem too special to be given here. It would be interesting to find a number $\equiv 3$ mod. 4 which could not equal the number of operators of order 2 in any group of order 2^m, or to prove the non-existence of such a number.

ON THE ARITHMETIC NATURE OF THE COEFFICIENTS IN GROUPS OF FINITE MONOMIAL LINEAR SUBSTITUTIONS.

BY DR. W. A. MANNING.

(Read before the American Mathematical Society, September 7, 1905.)

PROFESSOR MASCHKE[*] has proved (with a certain restriction) that the coefficients of finite linear substitution groups can, by proper transformations, be made rational functions of roots of unity. Professor Burnside[†] has also recently written on this subject. In this note it is proved that linear groups all of whose elements are finite monomial substitutions, that is, of the form

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[*] Maschke, *Math. Annalen* v. 50 (1898), p. 492.

[†] Burnside, *Proc. London Math. Society*, ser. 2, v. 3 (1905), p. 239.

$$x_i' = a_j x_j \qquad (i, j = 1, 2, \cdots, n),$$

(n is the number of variables) can be written so that all the coefficients are *roots of unity.*

THEOREM I. *Any non-vanishing element of the principal diagonal of a monomial substitution of finite period is a root of unity.*

Consider the kth power of a substitution S. If a lies in the principal diagonal of S, a^k occupies the same place in the principal diagonal of S^k. Hence a is a root of unity.

THEOREM II. *The product of two non-vanishing elements of the same or any two substitutions of a group* (G) *of finite monomial linear substitutions, which are symmetrically placed with respect to the principal diagonal, is a root of unity.*

The product of these two elements stands in the principal diagonal of the product of the two substitutions.

THEOREM III. *If G has no coefficient a_{ik} zero for all its substitutions, it can be transformed into another monomial group such that the non-vanishing element in the first column of every substitution of G becomes a root of unity.*

Since no element a_{ik} is zero for every substitution of G, we can choose $n - 1$ substitutions

$$A^{(2)} = (a_{ik}^{(2)}), \quad A^{(3)} = (a_{ik}^{(3)}), \cdots, A^{(n)} = (a_{ik}^{(n)}),$$

in which none of the coefficients

$$a_{12}^{(2)}, a_{13}^{(3)}, \cdots, a_{1n}^{(n)}$$

vanish. Transform G by the canonical substitution

$$x_i' = x_i / p_i \qquad (i = 1, 2, \cdots, n).$$

The transformed group (G') is monomial. If p_1 is an arbitrary root of unity, and if $p_2 = a_{12}^{(2)}$, $p_3 = a_{13}^{(3)}$, $\cdots$, $p_n = a_{1n}^{(n)}$, we have in place of $A^{(2)}, A^{(3)}, \cdots, A^{(n)}$ substitutions with p_1 as the only non-vanishing element in the first row of each. Now apply Theorem II to all the substitutions of G', and the present theorem follows.

THEOREM IV. *If G' has no coefficient everywhere zero the non-vanishing elements of every substitution are roots of unity.*

Consider the first column of a product AB. The elements of this column are obtained by multiplying the rows of A into the first column of B. Let A and B run through all the substitutions of G'. Every coefficient is seen to be the quotient of two roots of unity, that is, a root of unity.

Suppose that a certain coefficient a_{ik} vanishes in every substitution of G. We may assume that the variables of G have been so permuted that the $n-r$ last elements in the first row of all the substitutions of G are zero, and that no other row has more than $n-r$ elements which vanish for every substitution of G.

THEOREM V. *Every substitution of G is in the form*

$$\left[\begin{array}{c|c} N_1 & \begin{matrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{matrix} & N_2 \end{array}\right],$$

where N_1 and N_2 are monomial matrices, without further transformation.

There now are $r-1$ substitutions $A^{(2)}, A^{(3)}, \cdots, A^{(r)}$ in which the coefficients $a_{12}^{(2)}, a_{13}^{(3)}, \cdots, a_{1r}^{(r)}$ do not vanish. From the products

$$A^{(2)}B,\ A^{(3)}B,\ \cdots,\ A^{(r)}B,$$

$$a_{12}^{(2)}b_{2i}=0,\ a_{13}^{(3)}b_{3i}=0,\ \cdots,\ a_{1r}^{(r)}b_{ri}=0 \quad (i=r+1,\cdots,n),$$

where B is any substitution of G. Hence the last $n-r$ coefficients of the first r rows of all the substitutions of G are zero. Since these substitutions are monomial the first r elements in the last $n-r$ rows are also everywhere zero.

The group in the variables $x_1, x_2, \cdots x_r$ has by hypothesis no coefficients that are everywhere zero, so that for it Theorem IV holds.

We continue in this way with the group in the last $n-r$ variables, and finally have the

THEOREM VI. *The coefficients of a group of monomial linear substitutions of finite period may, by means of transformations which leave them monomial, be made roots of unity.*

STANFORD UNIVERSITY,
August 7, 1905.

A MODERN CALCULUS OF VARIATIONS.

Lectures on the Calculus of Variations. By OSKAR BOLZA. (The University of Chicago Decennial Publications, Second Series, Volume XIV.) The University of Chicago Press, Chicago, 1904. 8vo, xv + 271 pp.

THE calculus of variations is one of the very first of the old (or formal) developments of the infinitesimal calculus; it is one of the newest conquests of the modern (or critical) school. The history of the older calculus of variations is almost trite from reiteration; to select among many, the works of Moigno-Lindelöf, Diegner, Todhunter, Carll, Jordan, and (more recently and more perfectly) Pascal, have made known the achievements of the old school from Newton to Jacobi to mathematicians of all nations. The problems and the successes of the new school — the critical investigations — have been known to few until this century opened. In 1900 the first really modern treatise — Kneser's Lehrbuch — appeared. Without gainsaying the importance of that work and the fact that it opened the doors of modern research in the calculus of variations to the general mathematical public, it must be admitted that its style and arrangement are forbidding. The articles in the Encyklopädie (by Kneser and by Zermelo and Hahn) scarcely dispel the gloom of the Lehrbuch's unilluminated interior; but this was not to have been expected in encyclopedia articles, and the circumstances which necessitated two separate articles did not particularly conduce to public enlightenment on comparative values of various methods.

The book which is the subject of the present review is of interest to us because it is the first accurate and critical presentation of all the modern methods which is thoroughly readable, and because of the natural favorable prejudice of language and country. Apart from any such bias, however, Professor Bolza's treatise is surely admirable in many ways. Let us examine some of its notable features.

The preface is illuminating concerning much that has not been generally known, and the statements of the essential differences between the older theory and the new are worthy of recapitulation. They are: (1) the critical revision, by Weierstrass, Erdmann, Du Bois-Reymond, Scheefer, Schwarz, and others, of results previously obtained; (2) Weierstrass's introduction of the parameter representation; (3) Weierstrass's discovery of the fourth necessary condition and the sufficient conditions, with many subsequent allied contributions by others; (4) Kneser's extension to extremals of the theorems on geodetics; (5) Hilbert's *a priori* existence proof. That this is given at some length here and that the body of the book is often tinged with a historical coloring is doubtless due to the fact that these Lectures are a development of the Colloquium Lectures delivered by Professor Bolza at the Ithaca meeting of the AMERICAN MATHEMATICAL SOCIETY (August, 1901), which will be remembered with pleasure by many readers.

In the opening pages (§ 1, Introduction; § 2, Notation and terminology) one is made conscious of the rigorous nature of the discussions to be expected, but is reassured by the conciseness and clearness of statement, which go to make the book throughout easily readable to one otherwise familiar with exact reasoning. The reviewer would call attention to the precise definitions of interval, region, continuum, etc., on page 5. Some general agreement concerning these terms was and is needed, and the definitions here given may serve as a basis, even if they are not generally adopted verbatim. A difference of opinion is more tenable here, however, than in the demonstrations of later pages. Thus it might be better to reserve the name region for what is called a closed region, since the term continuum already connotes an absence of boundary points, and the word domain already serves as a generic term including closed regions, continua, and the intermediate cases. An interval, as defined here, is apparently always "closed."

The crucial article on the formulation of the problem (§ 3, page 9) leaves nothing to be desired. It should be noticed that maximum (or minimum) as here defined is that which is called "improper," in contradistinction to the case of "proper" maximum (or minimum). The usage in this matter must be settled by time. Although the present definition does not materially alter the conditions discovered by others under the other definition, it seems possible that it will change the nature of the necessary conditions which remain to be discovered (see page 100), and it would necessitate the recognition of any curve whatever as a minimizing curve in the case of integrals which are independent of the path. The use of the phrase "proper minimum" (see page 30, line 29) as distinguished from "minimum" may be misleading.

The next articles contain the two older classical derivations of the condition that the first variation should vanish (§ 4), Du Bois-Reymond's lemma which reduces the preceding condition to the *first necessary condition* (§ 5), and the Du Bois-Reymond-Hilbert proof of that condition without the assumption of the existence of the second derivative (§ 6). The presentations contain nothing essentially unfamiliar, except that the last proof is here presented for the first time in a separate work, but they are especially clear and the footnotes contain valuable remarks and references. The use of Euler's name instead of Lagrange's to designate the fundamental differential equation is novel and is apparently justified (footnote 3, page 22).

The miscellaneous remarks on the integration of Euler's equation in § 7 include several important matters, for example the proof of the existence of extremals in any direction not parallel to the Y axis (page 28), the case in which Euler's equation reduces to an identity (page 29), and the inverse problem (page 31). The formal side of the latter problem is outlined here in concise and satisfactory form, but it is unfortunately not reconsidered later after the development of the sufficient conditions.

The chapter closes with discussions of Weierstrass's lemma (§ 8), including the introduction of the E function and the concept of transversality; discontinuous solutions (§ 9); and (§ 10) the conditions at a boundary.

Chapter II is for the most part an account of the treatment of the so-called *second variation* by the methods of the old school. While the general lines of this work are familiar from older works, the nice distinctions and exact statements of this presentation are characteristic of the modern point of view and are not to be found in older works. We shall mention here only the following points: (1) the exact statement of Legendre's condition (page 47); (2) the exact statement of the assumptions (§ 13, pages 54–55) made in the proof of Jacobi's theorem; (3) the exact statement of Jacobi's condition (page 67). On the whole this chapter contains less that is modern, except in point of exactness, than Chapter I, but here also the notes and references are valuable even to one familiar with the older theory.

The study of the sufficient conditions in Chapter III formally opens the modern theory, though this would be impossible without the accurate revisions of the older theory in the previous chapters.

With respect to the author's view (page 70) that the concept of "weak" extrema, for which he has just proved that Jacobi's and Legendre's conditions are sufficient, is of only temporary importance, there may be differences of opinion. The distinction between "weak" and "strong" minima is of precisely the same kind as the distinction between "strong" minima and "absolute" minima, or as the distinction between isolated minima and absolute minima on a plane curve. In case no "strong" minimizing curve exists, for example, it seems interesting at least to discover the "weak" ones. Such is actually the case in Newton's famous problem on the surface of rotation of least resistance. Again, a "strong" minimizing curve is surely also a "weak" minimizing curve; hence we need seek for "strong" minimizing curves only among the "weak" minimizing curves.

Passing to the case of a strong minimum, Bolza first proves Weierstrass's (the fourth) necessary condition by means of Weierstrass's lemma. Then follows in §§ 19–20 the exquisite modern theory due to Weierstrass and Hilbert, resulting in Weierstrass's theorem and the first set of *sufficient* conditions. The matter presented is already fairly familiar to the American mathematical public through articles by Bolza himself, Osgood, and others, to which the most detailed references are given. But this is the first systematic treatment of the theory from all these points of view, and it is probably easier to compare the different methods here than in the isolated memoirs. It goes without saying that Kneser's treatment is much more one-sided here and in many other places, if for nothing else on account of the achievements of Hilbert and others, after the appearance of the Lehrbuch, which were considered to warrant the extra article in the Encyklopädie by Hahn and Zermelo. It seems to be conceded that the simplest and most elegant proof of Weierstrass's theorem is that based upon Hilbert's invariant integral, and subsequent treatments of the calculus of variations can probably afford to limit their discussion to a single proof. It might be advantageous to make even more use of Hilbert's principle than is indicated here. In particular the matter on page 266 (addenda) and that in footnote to page 175 indicate the possible important applications of this and similar methods.

The notation employed for the various conditions (I, II, II*a*, II*a*′, etc.) is somewhat novel but ceases to be confusing after careful examination, and the principal results are carefully stated in prominently lettered theorems. Thus the "Fundamental Theorem" (page 96), with the various appended notes (page 96 ; 1, page 99 ; and 1, p. 101), is sufficiently clear to dispel any confusion which might otherwise arise from such alternate statements as those on pages 98–101. The spirit of *c*), (page 98), shows the predilection of the author for the parameter representation of Weierstrass. Of course the omission of the equality signs from the necessary conditions to make them (apparently) coincide with the sufficient conditions is just as vital as any other difficult hiatus which might be suppressed as an "exceptional case," and a "fifth necessary condition" (see bottom of page 100) is just as indispensable in the case of parameter representation as it is here. But no such considerations are needed to establish the superiority of the parameter representation for a certain class of problems, nor could they possibly show a superiority for certain others — notably for certain problems of mechanics. The facts established in these pages (98–101) are nevertheless important and unquestionably rigorous, if we expressly repudiate any false inferences which a reader might draw upon his own account. Perhaps the theorem stated in footnote 1, page 101 is sufficiently important to have deserved a place in the text proper ; but the reviewer has elsewhere* insisted upon this point sincerely enough and with possibly an excess of fervor.

Finally, the table of all the conditions on pages 101 and 102 removes any ambiguity which may remain. This table is for all practical purposes the end of the consideration of the simplest problem — namely that of minimizing the integral

$$\int_{x_0}^{x_1} f(x, y, y')dx$$

between given fixed end points.

Dividing the chapters as he does, Bolza has included in this chapter (III, Sufficient conditions) a discussion of one of the many possible generalizations — probably the first which naturally suggests itself — namely the case in which the end points are variable. The presentation is by the "method of the differential calculus," which reduces the new problem to a combination of the preceding one and that of ordinary maxima and minima, which may be assumed as known (see footnote, pages 102–103). This fact renders this discussion somewhat less interesting as an independent theory and augments the importance of the previous problem. The reviewer therefore feels that a rather disproportionate fraction of the volume has been given to this matter, in view of the fact that it is taken up again twice. The results are interesting, particularly in connection with Kneser's theory (Chapter V), of which it forms almost an integral part, but in view of the limited size of the book and the important omissions mentioned below the three separate treatments might well have been condensed into a single one, say under Chapter V.

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* "Supplementary note on the calculus of variations," BULLETIN, vol. 9, No. 5 (February, 1905).

Chapter IV is an exposition of the theory of the calculus of variations in parameter form, according to the methods of Weierstrass. The author is very frank, and apparently entirely just in his estimate (footnote, page 115) of the advantages of the method. The asymmetric forms of the preceding chapters apply in reality, as the author remarks, to those problems in which the solution $y = f(x)$ is thought of as a function rather than as a curve, but this implies that the asymmetric forms are to be preferred not only in the theory of functions of a real variable, but also in numerous applications which are not primarily geometric. One is often deceived by the convenient employment of geometric terms into a belief that the essence of a problem is geometric when it is not so, and the number of problems in which the parameter representation is superior might easily be overestimated.

On the other hand there is no possible question of the immense advantage of the parameter representation for the precise treatment of geometric problems. This advantage is shared with absolute equality, however, by the similar treatment of any geometric application which arises in any branch of mathematics whatever. This fact has been recognized but a short time even in the very elements of geometry — even in the definition of a curve. It is for those nice considerations which become possible in an accurate critical revision of a theory, and not for the relatively crude work of discovery of new and general principles, however, that these forms are valuable even in geometry, and there is many a geometric problem which has not yet been stated in this manner. The calculus of variations is no more a geometric subject than is many another branch of mathematics, and the stimulus which the success of these methods in the calculus of variations will doubtless produce will surely cause the production of supplementary theories in these other subjects which will make applications to geometric problems as safe as are the applications of the results of this chapter. As a fitting example I might mention first of all the theory of differential equations, in which the ordinary theorems are no better suited to geometric applications than are those of the first three chapters of this book. A supplement to the theory of differential equations along the lines of the present chapter would be valuable and is indispensable to just the extent that the Weierstrass parameter representation is in the calculus of variations. As a matter of fact, on account of the perfect one-to-one correspondence of the real number system with the points upon a line and that of the assemblage of all pairs of real numbers with the points of a plane, we have been too unconscious of the discrepancy between curves on the one hand and equations of the form $y = f(x)$ on the other. It is perhaps not too much to say that geometric problems in general constitute by no means the nearest application of any well-known mathematical theory in the form in which it is at present usually given.

The development of the various conditions is wholly analogous to the work of previous chapters, and pages 115–146 and 153–156 involve few considerations which are *essentially* new; there are indeed new *facts*, but these are such as we should expect in any discussion in passing from non-homogeneous to homogeneous forms. In many cases omissions are made when the generalizations are rather obvious (see page 115, footnote 1). The following points deserve at least mention: (1) Weierstrass's form of Euler's equation (page 123) and the commentary upon it (page 124); (2) the proof (page 125) of the existence of an extremal in any direction; (3) the statement and proof of the necessary and the sufficient conditions (page 143); (4) the proof of the existence of a minimum "im Kleinen" (pages 146–147); (5) the conditions at a boundary (pages 148 ff.); and (6) the extension of the meaning of a definite integral (pages 156–163).

With regard to the statement of the sufficient conditions on page 143, the possibility of misunderstanding has already been mentioned. The reader should be careful to note that Bolza does not even imply that there is any one set of conditions given which is at once necessary *and* sufficient, though he seems to use those words. The four sufficient conditions stated on that page have nowhere been shown to be necessary, nor are they so (see footnote 2, page 148). While not wishing to criticize the real content of these statements, the reviewer feels that the actual gap between the necessary and the sufficient conditions should not only not be lost sight of, it should be emphasized and brought home by every possible device; the conscious or unconscious failure to do this in the past has been the bane of exact mathematical reasoning. For this reason the reviewer would object to the notation which apparently identifies two different conditions, one of which is necessary and the other sufficient, and he would object to the phraseology (in *d*, page 143) which bears out this false impression. As a matter of fact the necessary and the sufficient conditions are not appreciably nearer together than they are in the other — non-geometric — problem; they are not appreciably nearer than they have been for several years.

The proof of the existence of a minimum "im Kleinen," which follows Bliss, *Transactions of the American Mathematical Society*, volume 5 (1904), is very important in many applications and constitutes one of the several important and wholly new topics treated. It is regretable that no mention is made of Hilbert's concept of "Stückweise Variation,"* which is intimately connected with this topic and which has important applications.

The treatment of the extension of the meaning of a definite integral is also particularly noteworthy. The results of Weierstrass, Hilbert, Osgood, Peano and others are presented in their essentials and compared. The subject is again taken up in Chapter VII and forms an integral part of the author's excellent presentation of Hilbert's *a priori* existence proof.

The proof just mentioned is given in detail in Chapter VII, which we shall consider here on account of the connection mentioned above. The whole chapter is carefully written and contains the first accurate detailed proof of Hilbert's theorem. There is first a precise statement of the theorem and of the conditions under which it is to be proved. Then follows a supplementary investigation of extensions of the meaning of an integral, which together with the treatment just mentioned forms one of the important contributions of the book. I shall not enter into the details of the proof. It is rather long, but seems as concise as possible, and it actually proves, with apparently only necessary restrictions, all that was ever claimed for the Hilbert proof in the case of a simple integral. In its accuracy and generality this chapter really amounts to an original contribution to mathematics, since neither Hilbert nor anyone else has published in any place an accurate proof of the theorem in its most general form, although portions of the proof and special cases of it have been proved by Hilbert and others and the truth of the theorem rather generally accepted. Leaving aside any other merit, this chapter alone would make this book a pioneer work of the first importance.

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* The reviewer has elsewhere translated this as "limited variation." See BULLETIN, vol. 9, p. 12.

Chapters V and VI are devoted, respectively, to Kneser's transversal theory and the isoperimetric problem. Since the main features of both these chapters follow Kneser's Lehrbuch, they will be passed over rapidly.

Chapter V begins with the general explanation of Kneser's transversal theory as an extension of the ordinary theory of geodetic lines, and it is pointed out that this theory applies with peculiar force to the case in which the end points are variable. Kneser's principal theorems (on transversals, page 172; and on the envelope of a set of extremals, page 174) are given in § 33. It is shown in a note (page 175) that these theorems can be derived in a manner wholly analogous to Hilbert's original considerations which led to Hilbert's invariant integral theorem. These considerations are given in a note in the Addenda (page 266) in almost the same form in which Hilbert led up to his theorem in the course of lectures mentioned on pages 246 and 268. It should be mentioned that this course was formally entitled "Ausgewählte Capitel aus der Flächentheorie" and dealt with the calculus of variations *via* the problem of geodetic lines. The proof of the existence of a field (cf. Kneser, Lehrbuch, § 14) is made rigorous by the considerations of § 34, and the way is prepared for the introduction of Kneser's curvilinear coördinates. A number of important conclusions are stated in § 36, including Osgood's theorem "concerning a characteristic property of a strong minimum" (*Transactions of the American Mathematical Society*, volume 2, 1901). Kneser's proof of Weierstrass's theorem is compared with Hilbert's in § 37 and the unnecessary assumption in Kneser's proof is discussed in an enlightening manner. The chapter closes with a discussion of focal (or critical) points for the case of variable end-points.

The treatment of the isoperimetric problem follows Kneser's Chapter IV and Weierstrass's lectures. The essential facts —

Euler's rule, Meyer's law of reciprocity, the various necessary and sufficient conditions — are presented with the usual clearness and accuracy. In both these chapters which follow Kneser, the advantage over Kneser in clearness and conciseness is indisputable.

The reviewer is not one who would consciously conceal the faults, much less the errors, of a treatise under his criticism. But if there are serious errors in this book, they have escaped the reviewer either through ignorance or through insufficient scrutiny. The faults of a book are often matters of opinion, and I have pointed out a very few minor points in which I should venture to disagree with the notation or with the method of presentation. There are two other general criticisms of this nature. The arrangement of the matter into chapters was probably the result of careful consideration, but it seems that space might have been saved had the chapters been arranged according to subject matter rather than according to authors or the nature of the conditions to be proved. Thus Weierstrass's theorem, the case of variable end points, and several other matters are all considered in each of three distinct chapters. Again the repeated proofs of certain theorems by several methods consumes space, but this is made up for by the value of the comparisons which become possible. A future treatise will be spared this burden by reason of the existence of this book. As a result, perhaps, of the policy just mentioned, many important topics have been omitted altogether. For example, no attention is paid to the extension of the theory to double integrals, nor to the case in which higher derivatives enter in the integrand; and the consideration of the important inverse problem is restricted to the formal work on the two pages 31–32, which gives only the results due to Darboux.

These suggestions regarding arrangement and selection of topics do not alter the opinion already expressed that the treatment of the topics which are considered is excellent and that the book as a whole is an important contribution to mathematical knowledge not only in America but throughout the world. That Bolza's work is of international importance is shown by the speedy recognition it has found in at least one foreign quarter: the chapter on the calculus of variations in the second volume of Goursat's very recent Cours d'Analyse contains explicit acknowledgment of the indebtedness of its author to this treatise.

May this be a pioneer, not only in the subject which it represents, but also in the publication of accurate works on advanced mathematical topics in America.

E. R. HEDRICK.

THE UNIVERSITY OF MISSOURI,
COLUMBIA, MO.,
August, 1905.

TWO BOOKS ON ANALYTIC GEOMETRY.

The Elements of Analytic Geometry. 8vo. 424 pp. *Introduction to Analytic Geometry.* 8vo. 217 pp. By PERCEY F. SMITH and ARTHUR SULLIVAN GALE. New York, Ginn and Company, 1904.

It is a good sign for the university instruction in mathematics of the day that text-books are appearing more frequently in series than formerly. As a rule it means a better individual text-book, for it is the attempt of not one, but several members of the department collaborating to meet in the best possible way what actual experience has taught them to be the need of the student; and it certainly means a more organic, better coördinated mathematical curriculum in the institution which properly uses them. There is furthermore a considerable time economy for the student in the use of such a series. Take for instance the subjects: functions, graphs, partial fractions, and so forth, occurring in different lights in algebra, analytic geometry and calculus; considerable space and time can be gained in their second and subsequent treatments. Still another important gain is to be found in the clearing away for the student of all non-essential difficulties of a new subject, such as becoming accustomed to a new style, a new method of exposition, or a new notation. The books before us are two of a series called into being by needs at Yale University and now appearing under the editorial direction of Professor P. F. Smith.

In those of the works which are at present complete, the main object seems to have been to create drill books, which should be clear and minute in the exposition and analysis of method and rich in exercises, so that a course in one of them would leave the student a ready master of the more usual problems of the subject. If this is a correct conception of their aim, it has been admirably attained. Each bit of theory is immediately followed by a number of illustrative examples worked out in detail; these are followed by an analysis of the method used in the form of a set of two or three "working rules" for obtaining the required results; then comes a well-stocked and judiciously chosen set of exercises. Some of the special features of the books are well enumerated in the little announcements distributed by the publishers and need not be commented upon here. As the "Introduction" is built upon the first nine chapters of the "Elements" we may perhaps do best to touch first upon the topics common to both books, and then add a few words about the "Elements."

In looking over a new analytic geometry one usually turns straight to the chapter on locus and equation, as the vital part of the whole subject. In these books the chapter does not start by considering indeterminate equations and interpreting them geometrically as a series of points lying on a curve of given type, but proposes and attacks the problem of finding the equation of a given geometric locus at once. The "rule" formulated is fundamental and ought to be in every text book on the subject. An interesting point in this chapter is this: If a given equation the nature of whose locus it is required to discuss is seen to be a special case of a more general equation previously discussed, its locus is a special case of the general locus — this fact is made a principle, and called the "principle of comparison." It is an example of the detail with which the paths of thought are mapped out for the student. Indeed, if the books have any salient fault, it is this spirit of doing so much for the pupil, leaving him to do rather little thinking for himself and reducing nearly everything to mechanical rule.

Although the conic sections occupy a position avowedly secondary to the general methods of analytic geometry, they receive a systematic discussion in a separate chapter. The general equation of the conic is first derived in polar coördinates in terms of eccentricity and distance of focus from directrix. The forms of the special conics are then discussed, first the parabola, then the ellipse and hyperbola in parallel columns. The equations are then reduced to their simplest forms in rectangular coördinates. The more important properties of the conics and systems of conics are considered, and a general method is given for reducing and plotting the locus of the general equation of the second degree.

Tangents and normals receive discussion in a separate chapter, and the analytic geometry of space is treated in the two concluding chapters as a direct generalization of the developments for the plane.

The "Elements," while designed to meet the needs of the same class of students as the shorter book, constitutes a somewhat longer course, and is sufficiently rich in material to allow of considerable latitude of choice. A valuable chapter is added on equations in parametric form, also one on invariants of the quadric with respect to motions of the plane, one on euclidean transformations, one on inversion, one on poles and polars, including polar reciprocation, and the analytic geometry of space receives a much fuller treatment. The chapter on "Line and quadric" treats of tangent lines and planes, diametral and polar planes, and circumscribed and asymptotic cones.

The general appearance of the book is exceedingly attractive. A variety of styles of type has been skillfully used to make clear the different characters of various parts of the text, as theorem, proof, rule, and example. The figures are excellent, clear and distinct, and mention should be made of the halftones of models of quadric surfaces toward the end of the books, which will do a great deal in helping the student to a correct idea of these surfaces.

No estimate of a book can be quite satisfactory without a basis of class-room experience; but these two books certainly merit a trial, especially in institutions where the needs of the students are similar to those at Yale University.

O. D. KELLOGG.

PRINCETON, N. J.,
July 25, 1905.

NOTES.

THE German mathematical society held its annual meeting, in affiliation with the association of German scientists and physicians, at Meran, Tyrol, September 24–30. Twenty-seven titles were announced on the preliminary programme, including reports "On the introduction of the calculus in the gymnasia," by Professor E. CZUBER; "On partial differential equations of physics," by Professor W. WIEN; "New investigations of Riemann's ζ-functions," by Dr. P. EPSTEIN; "The status of the arithmetic theory of algebraic numbers," by Professor J. WELLSTEIN. A full report of the meeting will appear in a later number of the BULLETIN.

THE annual meeting of the French association for the advancement of science was held at Cherbourg, August 3 to 8, under the presidency of Professor A. GIARD. Professor M. D'OCAGNE was president of sections I and II, mathematics, mechanics and geodesy. Besides various papers on astronomical subjects, the following mathematical papers were read: "Difficulties in rational mechanics," by Professor P. APPELL; "The forms of functional singularities defined by a vanishing discriminant," by Professor CLARK; "On kinematic methods of determining the radius of curvature," by Professor D'OCAGNE; "Construction of magic squares for the first three degrees," by M. TARRY; "On the number of primes between 1 and n," by Professor LEBON; "Arithmetic spaces," by M. ARNOUX; "On the theory of polygonal numbers," by M. A. CABREIRA. After the meeting various excursions were made to points of interest in the vicinity of Cherbourg.

THE annual list of American doctorates published in *Science* presents for the academic year 1904–1905 324 names, of which 213 are credited to the sciences. The following 21 successful candidates offered mathematics as major subject (the titles of the theses are appended): O. P. AKERS, Cornell, "On the congruence of axes in a bundle of linear complexes"; R. B. ALLEN, Clark, "On hypercomplex number systems belonging to an arbitrary domain of rationality"; C. E. BROWN, Clark, "A study of the simple arithmetic processes"; Miss E. M. CODDINGTON, Columbia, "The historical development of pseudo-spherical surfaces"; W. B. FORD, Harvard, "On the problem of analytic extension as applied to functions defined by power series"; J. S. FRENCH, Clark, "On the theory of the pertingents to a plane curve"; J. N. GATES, Clark, "Cubic and quartic surfaces in four-fold space"; O. E. GLENN, Pennsylvania, "The determination of the abstract groups of order p^2qr, p, q, r being distinct primes"; U. S. HANNA, Pennsylvania, "The bitangentials of the plane quintic and plane sextic"; A. M. HILTEBEITEL, Princeton, "The problem of two fixed centers and certain of its generalizations"; H. E. JORDAN, Chicago, "Group characters of various types of linear groups"; H. G. KEPPEL, Clark, "The cubic three-spread ruled with planes in four-fold space"; R. B. McCLENON, Yale, "On simple integration with variable limits"; Miss A. M. McKELDON, Pennsylvania, "Groups of order 2^m that contain cyclic subgroups of orders 2^{m-1}, 2^{m-2}, 2^{m-3}"; T. E. McKINNEY, Chicago, "Concerning a certain type of continued fractions depending upon a variable parameter"; W. A. MANNING, Stanford, "Studies on the class of primitive substitution groups"; R. L. MOORE, Chicago, "Sets of metrical hypotheses of geometry"; J. C. MOREHEAD, Yale, "Numbers of the form $2^q - 1$ and Fermat's numbers"; H. B. PHILLIPS, Johns Hopkins, "Some invariants and covariants of ternary collineations"; A. W. SMITH, Chicago, "The symbolic treatment of differential geometry"; R. P. STEPHENS, Johns Hopkins, "I. On a curve of the fifth class, II. On a system of parastroids." The number of American doctorates in mathematics for each year from 1898 to 1905 is 11, 13, 11, 18, 8, 7, 14, 21.

IN the new edition of the Encyclopedia Americana about to be issued by the *Scientific American*, special attention has been given to mathematics. Over forty articles have been written for the encyclopedia by American mathematicians on their several specialties. Professor C. J. KEYSER is in editorial charge of this department of the encyclopedia.

THE publishing house of B. G. Teubner in Leipzig has announced a collection of mathematical models which will henceforth be sold by the firm. The collection at present consists of 59 models, selected from the exhibit prepared by Professor H. WIENER for the third congress of mathematicians at Heidelberg in 1904. It is to be increased and systematized and is intended for service as illustration in the class room rather than for original investigation.

IN its issue of September 15, 1905, the *Enseignement mathématique* begins the publication of the results of its inquiry regarding the method of work of mathematicians, noticed in the BULLETIN, volume 11, page 95. The first installment of eight pages is mainly concerned with answers to question 1*a*, "At what age and under what circumstances did you acquire a taste for mathematics?"

THE various foreign universities named below offer courses in mathematics during the winter semester of 1905–1906 as follows:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Analytic geometry, four hours; Theory of analytic functions, I., four hours; Applications of elliptic functions, 2 hours; Colloquium and seminar, two hours. — By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours; — By Professor F. SCHOTTKY: Theory of abelian functions, four hours; Theory of plane potential, two hours; Seminar, two hours. — By Professor J. KNOBLAUCH: Approximate calculation of definite integrals, one hour; Theory of elliptic functions, four hours; Analytic mechanics, four hours. — By Professor G. HETTNER: Infinite series, products and continued fractions, two hours. — By Professor L. LEHMANN-FILHÉS: Integral calculus, four hours; with exercises, one hour. — By Dr. I. SCHUR: Differential calculus, four hours; with exercises, one hour; Irrational numbers, one hour. — By Dr. E. LANDAU: Algebra, four hours.

UNIVERSITY OF BONN. — By Professor E. STUDY: Differential geometry, four hours; Introduction to the calculus of variations, two hours; Seminar, two hours. — By Professor G. KOWALEWSKI: Elements of analytic geometry of the plane and of space, four hours; with exercises, one hour; General theory of functions, three hours; with exercises, one hour; Seminar, one hour. — By Professor F. LONDON: Differential and integral calculus, II., four hours; with exercises, one hour; Synthetic geometry, two hours; Seminar, one hour.

UNIVERSITY OF BRESLAU. — By Professor J. ROSANES: Elements of determinants, two hours; Modern methods in analytic geometry, three hours; Seminar, one hour. — By Professor R. STURM: Theory of geometric correspondences, II, four hours; Line geometry, two hours; Seminar, two hours. — By Professor A. KNESER: Differential and integral calculus, five hours; Theory of functions, three hours; Seminar, one hour. — By Professor G. LANDSBERG: Algebraic exercises, two hours; Theory of equations, four hours; Introduction to the theory of algebraic functions, two hours.

UNIVERSITY OF FREIBURG. — By Professor J. LUROTH: Plane analytic geometry and differential calculus, five hours; Calculus of variations, three hours; Seminar, one hour. — By Professor L. STICKELBERGER: Theory of differential equations, four hours; Elements of the theory of numbers, two hours. — By Professor P. KÖNIGSBERGER: Elements of partial differential equations, three hours. — By Professor A. LOEWY: Algebraic analysis, four hours; Introduction to the mathematics of insurance, two hours; Seminar, one hour. — By Dr. K. SEITH: Projective geometry, two hours.

UNIVERSITY OF GIESSEN. — By Professor M. PASCH: Introduction to algebraic equations, four hours; Seminar, one hour. —By Professor E. NETTO: Differential and integral calculus, four hours; Analytic geometry of space, four hours; Seminar, two hours.— By Professor H. GRASSMANN: Geometric theory of functions with application to depiction, four hours; Graphical statics, three hours.

UNIVERSITY OF GREIFSWALD. — By Professor W. THOMÉ: Theory of potential, four hours; Calculus of variations, two hours; Seminar, two hours. — By Professor F. ENGEL: Analytic mechanics, I, four hours; Algebra, four hours; Differential equations and groups of transformations, two hours, Seminar, two hours. — By Professor K. TH. VAHLEN: Differential calculus, four hours; Theory of probabilities, two hours, with exercises, one hour.

UNIVERSITY OF HEIDELBERG: Professor L. KOENIGSBERGER: Analytic mechanics, four hours; Elliptic functions, two hours; Selected chapters of integral calculus (calculus of variations), two hours; Seminar, two hours. — By Professor E. VALENTINER: Planetary orbits, two hours. — By Professor M. CANTOR: Differential and integral calculus, four hours, with exercises, one hour; Political arithmetic, two hours. — By Professor K. KOEHLER: Analytic geometry of space, three hours. — By Professor K. BOEHM: Introduction to higher mathematics, three hours; Reading of memoirs, one hour.

UNIVERSITY OF KIEL. — By Professor L. POCHHAMMER: Theory of surfaces, three hours; Differential equations of one variable, three hours; Seminar, one hour. — By Professor L. HEFFTER (not yet announced): By DR. E. WEINNOLDT: Graphical statics, two hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Differential and integral calculus, four hours; Seminar, one hour. — By Professor A. MAYER; Analytic mechanics, four hours, with exercises, one hour. — By Professor O. HÖLDER: Elliptic functions, four hours; Galois's theory of algebraic equations, two hours; Seminar, two hours. — By Professor K. ROHN: Analytic geometry of space, five hours, with exercises, one hour; Descriptive geometry, II, two hours, with exercises, two hours; Seminar, two hours. — By Professor F. HAUSDORFF: Theory of transformation groups, three hours. — By Professor H. LIEBMANN: Theory of potential, two hours; Graphical statics, two hours.

UNIVERSITY OF MARBURG. — By Professor K. HENSEL: Algebra, four hours; Theory of surfaces and twisted curves, four hours; Seminar, two hours. — By Professor E. NEUMANN: Theory of functions, four hours; Analytic geometry of space, two hours; Seminar, two hours. — By Dr. F. v. DALWIGK: Statics, two hours; Applied mathematics, two hours; Higher questions of elementary mathematics, one hour. — By Dr. F. JUNG: Integral calculus, five hours.

UNIVERSITY OF ROSTOCK. — By Professor O. STAUDE: Differential and integral calculus, four hours; Theory of curves and surfaces, four hours; Seminar, two hours. — By Professor J. WACHSMUTH: Theory of potential, three hours.

UNIVERSITY OF STRASSBURG. — By Professor TH. REYE: Modern methods in the analytic geometry of space, three hours; Mathematics of elasticity of rigid bodies, two hours; Seminar, two hours. — By Professor H. WEBER: Differential and integral calculus, four hours; Encyclopedia of elementary mathematics, three hours; Seminar, two hours. — By Professor M. SIMON: History of mathematics in antiquity, two hours. — By Professor J. WELLSTEIN: Elliptic integrals, two hours; Determinants and matrices, three hours; Seminar, two hours. — By Professor E. TIMERDING: Plane analytic geometry with exercises, five hours; Graphical statics with exercises, three hours. — By Dr. P. EPSTEIN: Modern investigations in the theory of analytic functions, one hour; Seminar (with Professor Weber), two hours.

UNIVERSITY OF TÜBINGEN. — By Professor A. v. BRILL: Introduction to higher mathematics, four hours; Theory of algebraic curves, three hours; Seminar, two hours. — By Professor H. v. STAHL: Higher analysis, II (integral calculus), four hours; Partial differential equations, three hours; Seminar, two hours. — By Professor L. MAURER: Descriptive geometry, II, with exercises, three hours; Elliptic functions, two hours.

UNIVERSITY OF BASEL. — By Professor H. KINKELIN: Differential and integral calculus, I, three hours; Definite integrals, two hours; Partial differential equations, two hours; Algebraic analysis, two hours; Seminar, one hour. — By Professor O. SPEISS: Algebraic equations, two hours; Synthetic geometry, two hours; Elliptic functions, three hours.

UNIVERSITY OF BERN. — By Professor J. H. GRAF: Bessel functions, three hours; Definite integrals and gamma functions, three hours; Differential equations, two hours; Differential and integral calculus, two hours; Theory of functions, two hours; Mathematics of investments and insurance, two hours; Seminar, two hours. — By Professor K. OTT: Integral calculus, two hours; Plane analytic geometry, II, two hours. — By Professor G. HUBER: Celestial mechanics, two hours; Introduction to the theory of algebraic surfaces, three hours.

UNIVERSITY OF GENEVA. — By Professor C. CAILLER: Differential and integral calculus, three hours; Rational mechanics, three hours: Seminar, two hours. — By Professor H. FEHR: Algebra, two hours; Analytic geometry, two hours; Seminar, one hour. — By Dr. J. LYON: Algebraic theory of quadratic forms, one hour. — By Dr. R. DE SAUSSURE: Geometry of motion, two hours; Mechanics of fluids, one hour.

UNIVERSITY OF LAUSANNE. — By Professor M. AMSTEIN: Differential and integral calculus, I, six hours; II, two hours; with exercises, two hours; theory of functions, three hours. By Professor H. JOLY: Descriptive geometry, I, five hours; Analytic geometry, two hours; Synthetic geometry, two hours; Plane curves, two hours. — By Professor O. MAYOR: Rational mechanics, five hours; with exercises, one hour. — By Dr. C. JACCOTTET: Spherical harmonics, two hours.

ACADEMY OF NEUCHATEL. — By Professor L. ISELY: Calculus with geometric applications, three hours; Projective geometry, two hours. — By Professor L. GABEREL: Analytic mechanics, two hours.

UNIVERSITY OF ZÜRICH. — By Professor H. BURKHARDT: Elements of differential and integral calculus, four hours; Theory of potential, four hours; Seminar, two hours. — By Professor A. WEILER: Descriptive geometry, I, with exercises, four hours; Synthetic geometry, three hours; Analytic geometry, two hours. — By Dr. E. GUBLER: Algebraic analysis with exercises, two hours; Determinants, one hour; Spherical trigonometry, one hour; Geometric pedagogy, one hour.

BEGINNING on Friday, December 1, 1905, Professor V. F. BJERKNES, of the University of Stockholm, will deliver at Columbia University a course of lectures on the subject of "Fields of Force." These lectures are open, without charge, to all teachers and advanced students of physics who may desire to attend. They will be given on Fridays from 4 to 6 P. M. and on Saturdays from 10 to 12 A. M., December 1–23 inclusive. The topics to be treated in the several lectures are enumerated below:

December 1–2.—Elementary investigation of the geometric properties of hydrodynamic fields (with experiments).
December 8–9.—Geometric and dynamic properties of electromagnetic fields according to Maxwell's theory.
December 15–16.—Transformation of the hydrodynamic equations to forms which prove the analogy of hydrodynamic and stationary electromagnetic fields. Further development and discussion of this analogy.
December 22–23.—General conclusions. Remarks on methods of research and of instruction in theoretical physics. Supplementary lecture on the hydrodynamic fields of force in the atmosphere and the sea. Discussion of the fundamental problem of meteorology and hydrography.

The lectures of December 1–2 are introductory and require no special preparation. The most important lecture is that of December 15. Vector analysis as developed by Gibbs will be employed in the course.

PROFESSOR J. H. GRAF, of the University of Bern, and Professor R. v. LILIENTHAL, of the University of Münster, have been elected rectors of their respective institutions for the present academic year.

PROFESSOR R. HAUSSNER, of the technical school of Karlsruhe, has been appointed professor of mathematics at the University of Jena.

PROFESSOR F. SCHUR, of the technical school at Karlsruhe, has declined a call to the technical school at Charlottenburg, as successor to the late Professor Guido Hauck.

PROFESSOR O. TUMLIRZ, of the University of Czernowitz, has been appointed professor of mathematical physics at the University of Innsbruck, as successor to PROFESSOR K. F. J. EXNER, who has retired from active service.

PROFESSOR CH. MÉRAY, of the University of Dijon, has retired from active teaching with the title of honorary professor.

DR. J. BENDIXON has been appointed professor of mathematics at the University of Stockholm.

THE following have been appointed masters of mathematical conferences at the various French universities: ESCLANGON at Bordeaux, DULAC at Grenoble, CLAIRIN at Lille, AUTONNE at Lyon, LEBESGUE at Rennes, BOURGET at Toulouse.

MR. J. H. JEANS, M.A., of Trinity College, Cambridge, England, has been appointed to a professorship of applied mathematics at Princeton University.

AT Columbia University Mr. A. R. MAXSON has been appointed tutor in mathematics.

MR. G. C. DALY, Mr. H. W. POWELL, and Mr. L. P. SICELOFF have been appointed tutors in mathematics in the College of the City of New York.

DR. SAUL EPSTEEN has retired from the editorial staff of the *American Mathematical Monthly*, and has been succeeded by Professor O. E. GLEN, of Drury College.

AT Iowa College, Grinnell, Iowa, Professor S. J. BUCK has retired, having been professor of mathematics at that institution for over forty years. Assistant Professor W. J. RUSK has been promoted to a full professorship of mathematics.

DR. S. KRUSPER, professor emeritus of mathematics and geodesy at the technical school of Budapest, died July 2 at the age of 87 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Biermann (O.). Vorlesungen über mathematische Näherungsmethoden. Braunschweig, Vieweg, 1905. 8vo. 10 + 227 pp. M. 8.00

Bohler (O.). Ueber die Picard'schen Gruppen aus dem Zahlen-Körper der dritten und vierten Einheitswurzel. Zürich, 1905. 8vo. 102 pp. M. 3.00

Bucherer (A. H.). Elemente der Vektor-Analysis. Mit Beispielen aus der theoretischen Physik. 2te Auflage. Leipzig, Teubner, 1905. 8vo. 8 + 103 pp. Cloth. M. 2.40

Dalhuisen (A. A.). Over eenige aantallen van kegelsneden, die aan acht voorwaarden voldoen. Utrecht, 1905. 8vo. 8 + 86 pp. M. 3.00

Festschrift Adolf Wüllner gewidmet zum siebzigsten Geburtstage, 13. Juni 1905, von der Königl. Technischen Hochschule zu Aachen, ihren früheren und jetzigen Mitgliedern. Leipzig, Teubner, 1905. 8vo. 8 + 264 pp., 1 portrait, 8 plates. M. 8.00

Fiedler (W.). Analytische Geometrie der Kegelschnitte mit besonderer Berücksichtigung der neueren Methoden. Nach George Salmon frei bearbeitet. 6te Auflage. Zweiter Teil. Leipzig, Teubner, 1905. 8vo. 24 + 412 pp. M. 8.00

Fricke (R.). Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen zusammengestellt. 4te Auflage. Braunschweig, Vieweg, 1905. 8vo. 15 + 217 pp. M. 5.00

Geer (P. van). De herleving der mathematische wetenschappen. Leiden, 1904. 8vo. 45 pp. M. 1.00

Goursat (E.). Cours d'analyse mathématique. Vol. II: Fonctions analytiques; équations différentielles; équations aux dérivées partielles; éléments du calcul des variations. (En 3 fascicules.) Fascicule 2: Pp. 305–528. Paris, Gauthier-Villars, 1905. 8vo. (Cours de la Faculté des sciences de Paris.)

Grotendorst (N. C.). Beginselen der differentiaal- en integraalrekening. 2te, verbederde druk. Breda, 1904. 8vo. 10 + 514 pp. M. 13.00

Guldberg (A.). Sur les équations linéaires aux différences finies. 4to. (*Annales scientifiques de l'École normale supérieure* 21 (1905), pp. 309–348.)

Hagen (J. G.). Synopsis der höheren Mathematik. Vol. III: Differential- und Integralrechnung. 6te Lieferung: Pp. 321–384. Berlin, Dames, 1905. 4to. M. 5.00

Hill (G. W.). The collected mathematical works. Vol. I. Published by the Carnegie Institution of Washington, June, 1905. 4to. [With an introduction by H. Poincaré.] 18 + 363 pp., portrait. Cloth.

Koebe (P.). Ueber diejenigen analytischen Funktionen eines Arguments, welche ein algebraisches Additionstheorem haben. Berlin, 1905. 8vo. 32 pp. M. 1.50

Láska (W.). Wyklady nomografii. Opracowal i wydal F. Ulkowski. Lwów, 1904–1905. (autogr.)

Offerhaus (H.). Lineaire kegelsneestelses en weefsels. Groningen, 1905. 8vo. 8 + 100 pp., 2 plates. M. 3.00

Picard (E.). Traité d'analyse. 2e édition, revue et augmentée. Vol. II: Fonctions harmoniques et fonctions analytiques; introduction à la théorie des équations différentielles; intégrales abéliennes et surfaces de Riemann. Paris, Gauthier-Villars, 1905. 8vo. 15 + 587 pp. Fr. 18.00

Quitmann (E.). Ueber Minimallagen in ebenen Gebieten. Münster, 1905. 8vo. 45 pp., 1 plate. M. 1.50

Ruland (N.). Praktische Anleitung zum gründlichen Unterrichte in der höheren Mathematik. Ausführliche Auflösung der in Dr. Ed. Heis' Sammlung von Beispielen u. s. w. enthaltenen Aufgaben. Teil 3: Kettenbrüche, Teilbruchreihen, Permutationen, Kombinationen, Variationen, Wahrscheinlichkeitsrechnung, binomischer und polynomischer Lehrsatz, figurierte Zahlen, Gleichungen von höheren Graden, transzendente Gleichungen, Anwendung der Algebra auf Aufgaben aus der Geometrie, Physik und Astronomie. Zum Selbstunterrichte bestimmt. 2te, verbesserte Auflage. Bonn, Cohen, 1905. 8vo. 8 + 491 pp. M. 6.00

Rutgers (J. G.). Over differentialen van gebroken orde en haar gebruik bij de afleiding van bepaalde integralen. Utrecht, 1904. 8vo. 56 pp. M. 2.50

Salmon (G.). See Fiedler (W.).

Schoute (P. H.). Mehrdimensionale Geometrie. Teil 2: Die Polytope. Leipzig, Göschen, 1905. 12mo. Cloth. (Sammlung Schubert 36.) M. 10.00

Schubert (H.) Auslese aus meiner Unterrichts- und Vorlesungspraxis. Vol. I. Leipzig, Göschen, 1905. 12mo. M. 4.00

Tannery (J.). Introduction à la théorie des fonctions d'une variable. 2e édition, entièrement refondue. Vol. I: Nombres irrationnels; ensembles; limites; séries; produits; infinis: fonctions élémentaires; dérivées. Paris, Hermann, 1904. 8vo. 9 + 423 pp. Fr. 14.00

Ulkowski (F.). See Láska (W.).

Zaalberg (A. L.). Differentiaal-meetkundige eigenschappen van stralenstelses. Leiden, 1905. 8vo. 8 + 120 pp. M. 2.50

II. ELEMENTARY MATHEMATICS.

Baker (W. M.) and Bourne (A. A.). First algebra. London, Bell, 1905. 12mo. 186 pp. Cloth. 1s. 6d.

——With answers. 2s.

——Key to elementary algebra. London, Bell, 1905. 12mo. Cloth. 10s.

Benoit (P.). Essai d'une géométrie nouvelle. Saint-Dié, Weick, [1905]. 8vo. 159 pp.

Bidschof (F.) und Vital (A.). Fünfstellige mathematische und astronomische Tafeln. Zum Gebrauche für Mathematiker, Astronomen, Geographen und Seeleute zusammengestellt und mit Formelsammlungen versehen. Wien, 1905. 8vo. Cloth. M. 7.50

Böttcher (R.) und Sendler (R.). Raumlehre für Lehrerbildungsanstalten. Teil 2: Stereometrie und Trigonometrie. Resultate. Breslau, Handel, 1905. 8vo. 15 pp. M. 0.25

Borchardt (W. G.). Elementary algebra. With answers. London, Rivington, 1905. 12mo. 564 pp. Cloth. 4s. 6d.

BOURNE (A. A.). See BAKER (W. M.).

BRIDGETT (R. C.). An introduction to algebra. London, Blackie, 1905. 12mo. 96 pp. Cloth. 1s.

BUSH (W. N.) and CLARKE (J. B.). The elements of geometry. New York, Silver, Burdette, & Co., [1905]. 12mo. 12 + 355 pp. Half leather. $1.25

CLARKE (J. B.). See BUSH (W. N.).

DIEKMANN (J.). See KOPPE.

GÉOMÉTRIE. Cours moyen. Par une réunion de professeurs. Paris, Poussielgue, [1905]. 16mo. 191 pp. (Collection d'ouvrages classiques rédigés en cours gradués.)

GILDEMEISTER (T.). The multiplication tables. Chicago, Flanagan, [1905]. 12mo. 31 pp. $0.10

GRÉVY (A.). Trigonométrie, à l'usage des élèves des classes de seconde et première C et D et mathématiques A et B (programmes du 31 mai 1902). 3e édition. Paris, Vuibert et Nony, 1905. 16mo. 8 + 274 pp.

KOPPE und DIEKMANN. Geometrie zum Gebrauche an höheren Unterrichtsanstalten. Zweiter Teil der Planimetrie, Stereometrie und Trigonometrie. Ausgabe für Reallehranstalten. (19te Auflage.) 3te Auflage der neuen Bearbeitung von J. Diekmann. Mit zahlreichen Uebungen und Aufgaben. Essen, Baedeker, 1905. 8vo. 4 + 270 pp. Cloth. M. 2.40

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THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and twenty-fifth regular meeting of the American Mathematical Society was held in New York City on Saturday, October 28, 1905. The attendance at the two sessions included the following thirty members of the Society:

Professor D. P. Bartlett, Professor G. A. Bliss, Dr. W. H. Bussey, Professor F. N. Cole, Miss E. B. Cowley, Dr. W. S. Dennett, Professor L. P. Eisenhart, Professor T. S. Fiske, Mr. S. A. Joffe, Dr. Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Professor E. O. Lovett, Professor James Maclay, Dr. Max Mason, Professor H. B. Mitchell, Dr. C. L. E. Moore, Professor W. F. Osgood, Professor James Pierpont, Miss I. M. Schottenfels, Professor D. E. Smith, Professor P. F. Smith, Dr. C. E. Stromquist, Professor J. H. Tanner, Professor H. D. Thompson, Miss Mary Underhill, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor J. W. Young, Professor H. S. White.

The simultaneous meeting of the American Physical Society afforded opportunity for the cultivation of cordial relations. The luncheon in the interval between the sessions and an informal dinner in the evening was arranged to bring the members of the two Societies into closer communion.

At the meeting of the Mathematical Society President W. F. Osgood occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor O. P. Akers, Allegheny College; Dr. R. B. Allen, Clark University; Professor Ernesto Cesàro, University of Naples; Lieutenant Colonel A. J. C. Cunningham, London, Eng.; Miss M. E. Decherd, University of Texas; Mr. W. W. Hart, Shortridge High School, Indianapolis, Ind.; Mr. H. N. Olsen, Bethany College; Mr. F. H. Smith, Southwestern Christian College. Twelve applications for membership in the Society were received.

A list of nominations for officers and other members of the Council was adopted and ordered placed on the official ballot for the annual election at the December meeting. Dr. W. H. Bussey was appointed assistant secretary of the Society.

The President was authorized to appoint a committee to arrange for the summer meeting and colloquium for 1906. The members of this committee are Professors Maschke, Pierpont, P. F. Smith, H. S. White, and the Secretary. A committee was also appointed to audit the Treasurer's accounts for the current year.

The following papers were read at this meeting:

(1) Professor W. B. CARVER: "On the Cayley-Veronese class of configurations."

(2) Professor JAMES PIERPONT: "Multiple improper integrals."

(3) Dr. EDWARD KASNER: "On the geodesics passing through a given point of a surface."

(4) Professor H. S. WHITE: "Poncelet quadrilaterals on a curve of the third order and a conic."

(5) Dr. MAX MASON and Professor G. A. BLISS: "A problem of the calculus of variations in which the integrand function is discontinuous."

(6) Professor G. A. MILLER: "Groups generated by two operators which transform each other into the same power."

(7) Dr. BURKE SMITH: "Determination of associated surfaces."

(8) Professor L. P. EISENHART: "Certain triply orthogonal systems of surfaces."

Professor Carver's paper was presented to the Society through Professor T. S. Fiske. This paper was read by title as were also the papers of Professor Miller, Dr. Smith, and Professor Eisenhart. Professor Carver's paper was published in the October *Transactions*. Abstracts of the remaining papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. Professor Pierpont's paper was a continuation of his paper read at the April meeting of the Society. In that paper he developed the theory of outer multiple proper integrals. In the present paper the cases that either the integrand or field of integration become infinite are treated. The notions of cell and of division of space into cells developed in the former paper play a fundamental role in the present paper. In addition, the notions of uniform and regular evanescence of the singular integral, here introduced for the first time, enable the author to develop the theory of improper integrals with great simplicity.

3. If the geodesics passing through a given point O of a sur-

face are projected orthogonally upon the tangent plane at that point, a system of plane curves is obtained with, in general, inflexions at O. The only cases where the projected curve has third order contact with its tangent line arise (1) when the geodesic starts from O in a principal or asymptotic direction, or (2) when O is an umbilic. In the latter case there are three exceptional curves in the system having fourth order contact with their tangent lines. Dr. Kasner next considers the projection of the geodesics passing through a point O' in the neighborhood of O; this leads to an interesting correspondence analogous to that of conjugate directions. Incidentally, a number of geometric interpretations of quantities of the third order in the theory of surfaces are brought to light.

4. Under certain conditions a triangle can vary freely with its vertices sliding upon a plane cubic curve and its sides continually touching a curve of the second class. Professor White shows that in this case necessarily a quadrilateral varies in the same way, with its six vertices on the cubic. The analytic conditions, three in number, are found by means of results contained in his paper on "Twisted cubic curves that have a directrix" (*Transactions*, volume 4 (1903)).

5. In a medium in which the index of refraction is variable, a ray of light usually follows the curve on which the time of passage is a minimum. The time is expressible as a definite integral, so that the problem of the determination of the path is one belonging to the calculus of variations. If the ray passes from one medium to another the path will have an angle at the dividing surface, where the index of refraction is discontinuous. This suggests at once the subject of the paper of Dr. Mason and Professor Bliss: the study of the problem of the calculus of variations in the plane when the function under the integral sign is discontinuous along a given curve. Some of the well-known results with regard to necessary conditions apply at once. The more important parts of the paper are the discussions of the condition corresponding to Jacobi's in the usual problem, the construction of a field, and the derivation of *sufficient* conditions for the integral to be a minimum.

6. Let t_1, t_2 be any two operators which transform each other into the α power. Since their commutator, $t_1^{-1}t_2^{-1}t_1t_2=t_1^{\alpha-1}=t_2^{\alpha-1}$,

is commutative with each of these operators, the group G generated by t_1, t_2 is the direct product of its Sylow subgroups. From the same relations it follows that the orders of t_1, t_2 are divisors of $(\alpha - 1)^2$. In particular, two distinct operators cannot transform each other into their squares but they may be so chosen as to transform each other into any other power which is not zero. Professor Miller first considers the case when $\alpha - 1$ is a prime number and finds that there are just one abelian and one non-abelian group which are generated by such operators. In particular, the quaternion group is the only possible non-abelian group which is generated by two operators which transform each other into their third powers. This group may also be defined by the fact that it is generated by two non-commutative operators which transform each other into their inverses.

When $\alpha - 1 = p^m$, p being a prime, there are $\frac{1}{2}m(m + 1)$ non-abelian groups which are generated by two operators which transform each other into the α power. The orders of the two generators must be equal to each other and all of these groups are conformal with abelian groups except when $p = 1$ and $m = 1$. In this special case we arrive at the quaternion group, as noted above. As G is the direct product of its Sylow subgroups the case where $\alpha - 1$ is not a power of a prime is readily deduced from this case. Since the abelian Sylow subgroups present no difficulties, the author has confined his attention to the case where all the Sylow subgroups are non-abelian. In this case the two generators t_1, t_2 are of the same order, and the number of possible groups which may be generated by t_1, t_2 has been determined.

7. Dr. Smith develops a method for finding the associate of a given surface when it is referred to its asymptotic lines. The formulas developed enable one to write down the cartesian coordinates of the associated surfaces at once, after one has solved an equation of the Laplace type. It is also shown how one can obtain the equations of pairs of associated surfaces by solving an equation of the Laplace type and performing six quadratures.

8. In the April number of the *American Journal of Mathematics* Professor Eisenhart discussed surfaces with the same spherical representation of their lines of curvature as pseudo-

spherical surfaces; for the sake of brevity they are referred to as A-surfaces. In the present paper he discusses the triply-orthogonal systems in which one family of surfaces are of this kind and calls such a system an A-system. With each A-surface of such a system there is *associated* a pseudospherical surface, namely, the one with the given representation of its lines of curvature. It is shown that all of these associated surfaces form part of another triply orthogonal system; hence all the A-systems are obtained from pseudospherical systems by transformations of Combescure and Darboux. In the former paper transformations of A-surfaces were discovered which are generalizations of the transformations of Bianchi and Bäcklund for pseudospherical surfaces. The determination is made of the transformations changing all the A-surfaces of an A-system into other A-surfaces forming a family of a new A-system. The following particular A-systems are considered: 1), when the associated pseudospherical surfaces form part of a system of Weingarten; 2), when the orthogonal trajectories of the A-surfaces are plane, or the lines of curvature in one system upon these surfaces are plane; 3), when the orthogonal trajectories of the A-surfaces have constant curvatures; 4), when the lines of curvature in one system are spherical, or the transformed surface, under right angle, have this property; 5), when one of the other families of surfaces of an A-system contains only spheres; and 6), when the A-surfaces of an A-system are surfaces of Bianchi of the elliptic, parabolic or hyperbolic types.

F. N. COLE,
Secretary.

THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE eighth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of California, on Saturday, September 30, 1905. The following seventeen members were present:

Professor R. E. Allardice, Professor G. C. Edwards, Professor M. W. Haskell, Professor D. N. Lehmer, Professor A. O. Leuschner, Dr. J. H. McDonald, Dr. W. A. Manning, Professor G. A. Miller, Professor H. C. Moreno, Dr. B. L.

Newkirk, Professor C. A. Noble, Dr. T. M. Putnam, Mr. Arthur Ranum, Professor Irving Stringham, Mr. L. C. Walker, Professor A. W. Whitney, Professor E. J. Wilczynski.

Professor M. W. Haskell presided during the two sessions. The following officers were elected for the ensuing year: Professor R. E. Allardice, chairman; Professor G. A. Miller, secretary; Professors E. J. Wilczynski, D. N. Lehmer, G. A. Miller, programme committee.

The following papers were read at this meeting:

(1) Professor C. A. NOBLE: "Note on loxodromes."

(2) Dr. W. A. MANNING: "Groups in which more than half of the operators may correspond to their inverses."

(3) Professor M. W. HASKELL: "A new canonical form of the binary sextic."

(4) Professor A. O. LEUSCHNER: "On a new method of determining orbits."

(5) Mr. ARTHUR RANUM: "The representation of linear fractional congruence groups with a composite modulus as permutation groups."

(6) Professor E. J. WILCZYNSKI: "On a system of partial differential equations in involution."

(7) Professor G. A. MILLER: "The groups which contain only three operators which are squares."

(8) Professor R. E. MORITZ: "On logarithmic involution, the commutative arithmetic process of the third order."

(9) Professor L. E. DICKSON: "The abstract group simply isomorphic with the general linear group in an arbitrary field."

(10) Professor L. E. DICKSON: "The abstract group simply isomorphic with the symmetric group."

(11) Professor M. W. HASKELL: "On a class of covariants of order n which give rise to a birational transformation."

In the absence of the authors the papers by Professors Moritz and Dickson were read by title. Abstracts of all the papers are given below. The abstracts are numbered to correspond to the titles in the list above.

1. If the sequence of meridians and of parallels of latitude on a surface of revolution be so determined as to cover the surface with a network of similar rectangles, the diagonals of these rectangles will be loxodromes. The determination of the appropriate sequence involves a simple differential equation, the solution of which leads, by quadrature, to the finite equation

of the ∞^2 loxodromes on the surface. Professor Noble's paper is a contribution to the methods of determining these loxodromes but does not claim any new results.

2. Professor Miller has called attention to the fact that a group is abelian if more than three fourths of its operators may be made to correspond to their inverses. Dr. Manning shows that if just three fourths or just two thirds the operators of a group may correspond to their inverses, certain groups of comparatively simple composition are determined. If more than two thirds, then three fourths or all the operators of the group may be made to correspond to their inverses.

3. It is well known that the binary sextic can be expressed as the sum of four sixth powers in an infinity of ways, but not much use has been made of such a canonical form owing to its lack of definiteness. Professor Haskell shows that an assumed relation between the four linear forms involved gives rise to a number of useful canonical forms. For instance, the reduction is possible in two ways if the linear forms in question be equianharmonic, in three ways if they be harmonic, in one way if a certain simultaneous covariant of their product and of the given binary sextic vanishes. In this last case the product is a covariant of order four and degree five.

4. Professor Leuschner's paper was very closely related to the one presented at the last meeting of this Section. It indicated some further applications of the "short method" and gave instances of the amount of time that could be saved by means of it. The application to the determination of the orbit of the seventh satellite of Jupiter was given in some detail.

5. The congruence group of linear fractional substitutions in one variable, whose modulus m is composite, is represented by Mr. Ranum as a transitive permutation group on the totality of symbols of the form $[a, b]$, or a/b, such that the greatest common divisor of a, b, and m is 1, and the symbols $[a, b]$ and $[ka, kb]$ are identical. If

$$m = \prod_{i=1}^{i=n} p_i^{a_i},$$

then the number of symbols is

$$m\prod_{i=1}^{i=n}\left(1+\frac{1}{p_i}\right).$$

Every separation of m into factors r and s prime to each other gives rise to two sets of imprimitive systems of the group, viz., the systems of symbols which are congruent to one another, mod r and mod s, respectively. Moreover, if m contains a power of a prime higher than the first ($m = p^a r$), then there are sets of imprimitive systems of symbols which are at the same time congruent to one another, mod r and mod p^k, for every value of k from 1 to $\alpha - 1$. Corresponding to every set of systems of imprimitivity there is an invariant subgroup for which they are sets of intransitivity.

6. In general the simultaneous system of partial differential equations

$$(1)\qquad \begin{aligned} \frac{\partial^2 z}{\partial x^2} &= a\frac{\partial^2 z}{\partial x\partial y} + b\frac{\partial z}{\partial x} + c\frac{\partial z}{\partial y} + dz, \\ \frac{\partial^2 z}{\partial y^2} &= a'\frac{\partial^2 z}{\partial x\partial y} + b'\frac{\partial z}{\partial x} + c'\frac{\partial z}{\partial y} + d'z \end{aligned}$$

has four linearly independent solutions, say $z_1, \cdots, z_4$. Upon this fact may be based a projective differential geometry of surfaces. If however, certain relations are satisfied between the coefficients, the common solutions of the two equations may depend upon an arbitrary function. This is the case in which the two equations are said to be in involution. Professor Wilczynski examines this exceptional case and finds that it corresponds to the case of developable surfaces. The projective differential geometry of all non-developable surfaces may therefore be based upon the invariant theory of system (1).

7. Every group besides the abelian group of order 2^a and of type $(1, 1, 1, \cdots)$ contains at least two operators which are squares of other operators in the group. If there are only two such operators they constitute an invariant subgroup and the corresponding quotient group contains only one operator which is a square. All the groups which satisfy this condition have been determined. Professor Miller's paper is devoted to a complete determination of the groups involving just three operators which are squares. The main results may be stated as follows: All these groups are direct products of the abelian

group of order 2^γ and of type $(1, 1, 1, \cdots)$ into the cyclic group of order 3, the symmetric group of order 6, or a group of order 2^β. When $\beta = 4$, there are two such groups of order 2^β. When $\beta = 5$, there are three; when $\beta > 5$ there are four. In the last case, two of the groups have a commutator subgroup of order two while the other two have a commutator subgroup of order four. Each of these four groups has exactly $2^{\beta-2} - 1$ subgroups of order $2^{\beta-1}$. The paper has been offered to the *Transactions* for publication.

8. Eisenstein, Woepcke, Paugger, Gerlach and De Morgan have attempted to set up manageable processes of higher orders than the third by following analogies of form or structure of the ordinary processes. All these attempts were unsuccessful, except De Morgan's, whose definition seems however arbitrary and artificial. Professor Moritz points out that simplicity of form, or unity in structure, however desirable, are less essential than unity in fundamental laws. The principle of permanence, which has governed the extension of the number body, should be recognized in the extension of the number processes. This principle demands that ordinary involution, a^b, be replaced by the process $a^{\log b}$ which may be appropriately termed logarithmic involution. From this in turn the principle of permanence leads directly to the higher processes of De Morgan. Logarithmic involution obeys all the laws of the two lower processes and could be made to serve the purposes of ordinary involution. The paper is accompanied by two tables, one for performing logarithmic involutions directly, the other by means of logarithms. It will be offered to the *Annals of Mathematics* for publication.

9. It is shown in the paper of Professor Dickson that the group of all n-ary linear homogeneous substitutions of determinant unity in any given field F is simply isomorphic with the abstract group generated by the operators $B_{ij\lambda} (i, j = 1, \cdots, n;\ i \neq j;\ \lambda$ ranging over $F)$, subject to the generational relations $B_{ij\lambda} B_{ij\mu} = B_{ij\lambda+\mu}$, $B_{ij\lambda}$ commutative with $B_{kl\mu}$, $B_{ik\mu}$, $B_{kj\mu}$

$$B_{ij\lambda}^{-1} B_{jk\mu}^{-1} B_{ij\lambda} B_{jk\mu} = B_{ik-\lambda\mu}, \quad B_{ij\lambda} B_{ji\mu} B_{ij\nu} = B_{ji\mu\nu k^{-1}} B_{ijk} B_{ji\lambda\mu k^{-1}}$$

for $k \equiv \lambda + \nu + \lambda\mu\nu \neq 0$, where i, j, k, l, are any distinct integers $\leqq n$ while λ, μ, ν are any marks of F subject to a condition only in the last set of relations. In the concrete group,

$B_{ij\lambda}$ represents the substitution which alters only ξ_i, replacing it by $\xi_i + \lambda\xi_j$.

Another symmetric definition employs the generators $B_{i\,i+1\,\lambda}$ and $B_{i+1\,i\,\lambda}$ ($i = 1, 2, \cdots, n - 1$ variables).

As corollaries are deduced symmetric definitions of the abstract groups isomorphic with the linear fractional and linear non-homogeneous groups on $n - 1$ variables.

10. In his second paper Professor Dickson points out that the symmetric group on n letters is simply isomorphic with the abstract group generated by the operators $T_{ij} \equiv T_{ji}$ ($i, j = 1, \cdots, n$; $i \neq j$) subject to the relations $T_{ij}^2 = I$, $T_{rk}^{-1} T_{rs} T_{rk} = T_{sk}$, T_{ij} commutative with T_{kl} when i, j, k, l, are all distinct. The proof is immediate. Another symmetric definition by fewer generators is due to Professor Moore; *Proceedings of the London Mathematical Society*, volume 28, page 357.

11. Professor Haskell has shown before that there is a birational transformation between the coëfficients of a binary cubic and of its cubicovariant. In seeking for a generalization of this relation he finds that the binary n-ic, if $n = 4m - 1$, possesses a covariant of order n whose coefficients are then related to those of the ground form; if $n = 4m$, there is such a relation provided a certain invariant vanishes; while for $n = 4m + 1$ or $n = 4m + 2$, there is no covariant bearing this property.

The next meeting of the Section will be held at Stanford University on February 24, 1906. The newly elected officers will begin their term of office with this meeting.

G. A. MILLER,
Secretary of the Section.

NOTE ON LOXODROMES.

BY PROFESSOR C. A. NOBLE.

(Read before the San Francisco Section of the American Mathematical Society, September 30, 1905.)

A LOXODROME is a curve on a surface of revolution which meets the meridians at a constant angle. If the meridians and the parallels of latitude on such a surface can be so selected as to constitute a network of similar infinitesimal rectangles, the diagonals of these rectangles will give loxodromes.

Let the surface be given by

$$x = p(u) \cos v, \quad y = p(u) \sin v, \quad z = q(u).$$

The meridians are the curves $v = \text{const.}$, the parallels the curves $u = \text{const.}$

Let the network be conditioned by prescribing for u and v the increments

$$du = \alpha(u) \cdot \epsilon, \quad dv = \epsilon,$$

where ϵ is an infinitesimal and $\alpha(u)$ a function to be determined.

Two consecutive rectangles of the network, *e. g.*, those with corresponding vertices at (u, v) and $(u + du, v + dv)$ respectively, will be similar if the angle between the two diagonals of the network which pass through (u, v) is equal to the corresponding angle between the two diagonals of the network which pass through $(u + du, v + dv)$. The condition for this equality, to within second order infinitesimals, is

$$\frac{\alpha'(u)}{\alpha(u)} = \frac{p'(u)}{p(u)} - \frac{\sigma'(u)}{\sigma(u)} \quad (\sigma(u) = \sqrt{[p'(u)]^2 + [q'(u)]^2}),$$

where accents denote differentiation.

This relation yields

$$\alpha(u) = \mu \cdot \frac{p(u)}{\sigma(u)} \qquad (\mu = \text{const.}).$$

The diagonals passing through (u, v) are given by

$$dv = \frac{du}{\alpha(u)},$$

that is, by

$$v = A \int \frac{\sigma(u)}{p(u)} du + B \qquad (A, B = \text{const.}).$$

This, then, is the finite equation of the ∞^2 loxodromes on a surface of revolution.

For the spheroid formed by the revolution about the z axis of the ellipse,

$$x = a \cos u, \quad z = b \sin u,$$

the above equation becomes

$$v = A\left\{\tfrac{1}{2}e\log\left[e^2\cos^2 u - \frac{1+e^2}{2} + e\sqrt{1-(1+e^2)\cos^2 u + e^2\cos^4 u}\right]\right.$$

$$\left. + \tfrac{1}{2}\log\left[\sec^2 u - \frac{1+e^2}{2} + \sec^2 u\sqrt{1-(1+e^2)\cos^2 u + e^2\cos^4 u}\right]\right\} + B$$

where e denotes the eccentricity of the ellipse. For the unit sphere this reduces to

$$v = A \log\tan\left(\frac{u}{2} + \frac{\pi}{4}\right) + B.$$

It is obvious, and well known, that if the sphere be stereographically projected upon the plane of the equator, the loxodromes will map into logarithmic spirals. The equation of these spirals can be found readily, with the aid of the principle that every conformal mapping of one surface upon another transforms the ∞^2 minimal curves of the one surface into the ∞^2 minimal curves of the other. The minimal lines on the unit sphere are given (Scheffers: Einführung in die Theorie der Flächen, page 64) by

$$\bar{u} = \log\tan\left(\frac{u}{2} + \frac{\pi}{4}\right) + iv = \text{const.},$$

$$\bar{v} = \log\tan\left(\frac{u}{2} + \frac{\pi}{4}\right) - iv = \text{const.}$$

If these lines are selected as parameter lines, the equations of the sphere become

$$x = \frac{\bar{u} + \bar{v}}{\bar{u}\bar{v} + 1}, \qquad y = \frac{-i(\bar{u} - \bar{v})}{\bar{u}\bar{v} + 1}, \qquad z = \frac{\bar{u}\bar{v} - 1}{\bar{u}\bar{v} + 1},$$

and the loxodromes on the sphere become

$$\cos^{-1}\frac{\bar{u} + \bar{v}}{2\sqrt{\bar{u}\bar{v}}} = A\log\sqrt{\bar{u}\bar{v}} + B.$$

Stereographic projection establishes the relations (Scheffers, page 80)

$$\bar{x} = \tfrac{1}{2}(\bar{u} + \bar{v}), \quad \bar{y} = -\tfrac{1}{2}i(\bar{u} - \bar{v})$$

between the coördinates $\bar{x}$, $\bar{y}$ in the plane of projection, and the minimal parameters $\bar{u}$, $\bar{v}$ on the sphere. These relations yield, as the equation of the curves into which the loxodromes project,

$$\cos^{-1}\frac{\bar{x}}{\sqrt{\bar{x}^2+\bar{y}^2}} = A\log\sqrt{\bar{x}^2+\bar{y}^2}+B$$

or, in polar coördinates,

$$\rho = \alpha e^{\beta\theta} \qquad (\alpha,\ \beta = \text{const.}).$$

The case of the oblate spheroid is interesting, in that the loxodromes on it project stereographically into the same spirals as do the loxodromes on the sphere which is tangent to the spheroid along the equator.

BERKELEY, *October*, 1905.

STOLZ AND GMEINER'S FUNCTION THEORY.

Einleitung in die Funktionentheorie. Abteilung I. By OTTO STOLZ and J. ANTON GMEINER. Leipzig, B. G. Teubner. 1904. vi + 242 pp.

THE Funktionentheorie of Stolz and Gmeiner is a working over of certain parts of Stolz's Allgemeine Arithmetik which do not appear in the new Stolz and Gmeiner's Theoretische Arithmetik. The two new books are evidently to be thought of together as a single course in the elements of analysis. The contents of the Theoretische Arithmetik correspond in a general way to sections 1 to 7, part of 10, and most of 11 of volume I, and of sections 1, 2, 6, and part of 5 of volume II of the Allgemeine Arithmetik. The Theoretische Arithmetik begins with the theory of whole numbers; then, after discussing the system of rational numbers, positive and negative, develops for the real and ordinary complex numbers the theory of addition, subtraction, multiplication, division, exponents and logarithms. In the course of this development appears a short discussion of complex numbers of n units, including quaternions, some geometric applications, and the fundamental theorems on infinite series.

The Funktionentheorie, on the other hand, corresponds to sections 9, 11, and part of 10 of volume I, and to sections 3, 4, 7, 8, and part of 5 of volume II of the Allgemeine Arithmetik. The first Abteilung, which is before us, contains the general subjects of functions of one and of several variables, and the special subjects of rational integral functions and of power series in one or two variables. In the second part the authors promise to treat criteria for the convergence and divergence of series, Weierstrass's theory of analytic functions of one variable, circular functions, infinite products, finite and infinite continued fractions.

Function theory is understood in a different sense from that of mere continuity considerations. For example, Abschnitt IV, on rational integral functions, contains an extensive portion of what is ordinarily called algebra — divisibility, highest common factor, etc. On the other hand, the operations of infinitesimal calculus are excluded. $D_x f(x)$ is indeed defined for rational integral functions and for power series, but by an algebraic definition, and not by a direct limiting process.

The intention of the authors seems not to be to develop the exceedingly general theorems that are possible in function theory and then to get the applications to particular functions from these as special cases. They rather aim to complete the theory of the simpler functions alone, proving the general theorems only where they are needed for the cases of rational functions or of analytic functions. The exposition of real function theory in the ordinary sense goes but little beyond the definitions.

The argument in favor of this mode of exposition is of course that in order to understand the "ordinary functions" one does not need or wish to give his attention to the refinements necessary for the general theorems. But there are certain general theorems, treated in a very cursory way by our authors, to which this argument does not seem to apply with great force. For example, in proving Weierstrass's theorem on double series (page 206) they find it necessary to go clear back to first principles instead of referring to a theorem about the equality of $L_x L_y f(x, y)$ and $L_y L_x f(x, y)$ as they might easily have done if their analysis of this equality had been at all profound. The effect in this case is that the reader is subjected to practically all the difficulties of the general theorem in proving the special case at a time when his interest is in the elegant and rapid development of the theory of analytic functions rather than in the niceties of real function theory. Another consideration is that the demonstration of a theorem like the one about

$L_x L_y f(x, y) = L_y L_x f(x, y)$ in a special case is less likely than in the general case to bring out the issue essentially involved and is more likely to impress the reader as a mere cloud of symbols.

Another instance in which it seems to the reviewer that there would be a gain in using general principles of broader scope is the class of theorems expressing connections between the behavior of a function on an interval and at a point or points. Examples are the theorem of uniform continuity and the theorem that if a function $f(x)$ has on an interval a least upper bound b, there will be a point in every neighborhood of which $f(x)$ has the least upper bound b. The method of proof of Stolz and Gmeiner is the familiar one, to divide the interval on which a function is supposed to possess a certain property into e ($e \geqq 2$) equal parts. On at least one of these parts the function has the same property. The process is then repeated, and so on.

To go through this soporific ritual whenever we meet one of these theorems seems to the reviewer just as much in the spirit of modern mathematics as it would be actually to multiply out $(a + b)^n$ whenever we meet it in a special case, instead of using the binomial theorem. The modern way is to prove once for all a general theorem about classes of segments which may be cited in the special cases. One form of such a proposition is the Heine-Borel theorem to which we have referred in a former number of the BULLETIN.* Another useful form is the theorem of Cantor which is to be found on page 48, volume II of the Lezioni di analisi infinitesimale of G. Peano. Of course it will be objected that these theorems about classes of segments, like the theorems about point sets, are very abstract. But one must always have recourse to abstractions if he wishes to avoid tiresome repetitions of the same idea.

Another instance of the tendency of our authors to avoid the more general concepts is that the definition of continuity at a point (page 36) requires that there shall be an interval including the point upon every point of which the function is defined. So far as the reviewer observed there was no theorem whose demonstration was facilitated by this restriction. Another curiosity is in the definition of a variable (page 1): "Unter einer reellen Veränderlichen versteht man ein Zeichen, das unbegrenzt viele reelle Zahlen *deren jede völlig bestimmt sein muss,* bedeuten kann." The phrase which we have italicized appears again on page 114 in the definition of a complex variable and in other places. But what sort of a thing is a number which is not fully determined? We cannot believe that any clearness is gained by introducing such redundancies.

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* The Heine-Borel Theorem, BULLETIN, vol. 10, p. 436.

In "geometrically representing" a real function (pages 61–62) the authors believe it necessary to limit attention to such graphs as are composed of a finite number of straight or of convex curved segments. This, because only such curves can be constructed by mechanical means.

Throughout the book we get the impression that geometry is thought of, not as a science with the same logical ideals as arithmetic, but as a more or less unorganized mass of intuitions. For the purposes of analysis these intuitions are connected mainly with the notion of marking on a blackboard or a piece of paper. A corollary of this comment is that questions of topology (or analysis situs) are avoided as far as possible. Where it is not possible to avoid them, as in the complex variable theory, geometric notions are used without anything like the keen analysis that is expended on the corresponding arithmetical ideas. Compare pages 115, 116, 199.

The attitude toward geometry here complained of, and to a large extent the other objects of complaint, are not characteristic of this book alone, but of the whole extreme arithmetizing tendency in analysis. The criticism is made from the point of view of those who believe that rigor is not much more difficult in geometry than in arithmetic and that by the use of the generalizations and the illuminating language of geometry a much greater elegance can be obtained than by a strictly arithmetical method. It is made moreover with a thorough realization that the points attacked can all be strongly defended.

Commendation of this particular book is the easier on account of the elegance of its literary style. It is well arranged both in its large and in its small divisions. The language is generally clear and simple — though in some of the ϵ and δ arguments the interdependences of the various epsilons are not sufficiently indicated. The references to the sources and history of the theories treated are satisfactorily complete without being too verbose. The definitions are followed by good illustrative examples. Each of the five Abschnitte is followed by an interesting collection of Übungen. There is in general such an effect of dignity and maturity of thought that one feels that the book could be but little improved without changing its whole method.

An interesting question very well handled is the use of the symbols ∞, $+\infty$, $-\infty$. With the real number system can be associated *two* extra symbols, $+\infty$ and $-\infty$, having order relations with the rest and serving as the upper and lower bounds of the number scale (pages 1, 3). With the number system can be associated *one* extra symbol, ∞, having operational relations with the rest (page 6). That is, if the variable x be represented by pairs of numbers $\{x_1, x_2\}$ such that x is represented by the class $[\{x_1, x_2\}]$ for which $x_1/x_2 = x$, then one excludes from consideration $\{0, 0\}$ and represents the class $\{x, 0\}$ by ∞. The symbol ∞ is thus essentially without algebraic sign. On page 10 we find ∞ as a value of a function in case of a finite number of infinite discontinuities and on page 37 we find the notion of continuity at ∞.

The book is well printed, as the books of this publisher generally are. We have noted only three typographical errors. They should be corrected as follows:

Page 78, 18th line from bottom read $\phi(y)$ for $\phi(x)$.

Page 88, 11th line from bottom read $a + \delta_1$ for $a < \delta_1$.

Page 142, 14th line from bottom read $(n - r - 1)$ for $(n - p - r - 1)$.

OSWALD VEBLEN.

PRINCETON, N. J.,
September, 1905.

CESÀRO-KOWALEWSKI'S ALGEBRAIC ANALYSIS AND INFINITESIMAL CALCULUS.

Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung. Von E. CESÀRO. Deutsch von G. KOWALEWSKI. Leipzig, B. G. Teubner, 1904. 8vo. 6+894 pp.

THE above work, which was translated from the author's manuscript, is a revision of the Analisi Algebrica (1894) and the Calcolo Infinitesimale (1897). While the text is somewhat changed, the revision consists mostly in rearrangement partly made necessary by publishing the two books as one. The book is the outgrowth of lectures by Cesàro on algebraic analysis given simultaneously with a course on analytic geometry. The student is referred to such writers as Dini, Weber, Stolz, Jordan for more extended discussions of the principles, but the author has given great weight to the application of the principles developed, and there results an attractive and usable book on the calculus.

The book is attractive in appearance. Headings are given so that reference to any topic can be made readily, but this might have been aided by a full table of contents instead of a single page. Throughout the work enough references are made to the authors of noted theorems and to original articles to give the student a fairly good idea of the historic development of the subject.

We proceed to give a brief outline of the subject matter. The work is divided into seven books.

Book I. Determinants, linear and quadratic forms. — A short discussion of invariants and the reduction of a binary form to its canonical form is given.

Book II. Irrational numbers, limiting values, infinite series and products. — Under the title limiting values the limit of the last term of a sequence of numbers is discussed, the first part being devoted to the existence of the limit and the second part to its calculation. In this chapter the student makes his first acquaintance with the ϵ proof, first used in some simple demonstrations, which, however, bring out clearly the purpose for which the quantity ϵ was introduced. After this discussion of limiting values, series are taken up and the criteria for convergence are discussed at some length and some well chosen examples given. The book closes with a short discussion of infinite products, the necessary and sufficient condition in order that an infinite product converge toward a finite limit (not zero) are derived. The principal illustration used is the gamma function, which is defined as the infinite product

$$\Gamma(x+1) = \frac{\left(\frac{2}{1}\right)^x}{1+\frac{x}{1}} \cdot \frac{\left(\frac{3}{2}\right)^x}{1+\frac{x}{2}} \cdot \frac{\left(\frac{4}{3}\right)^x}{1+\frac{x}{3}} \cdots$$

instead of the definite integral as usual.

Book III. Theory of functions. — The following subjects are discussed: functions of one real variable, convergence of a function toward a limit, derivative, continuity, development of functions in series, and functions of several variables. Functions are defined, according to Dirichlet, as single valued functions. Throughout this book considerable importance is attached to the mean value theorem. A great many theorems usually proved in American text books by means of Taylor's series are proved here by the use of the mean value theorem. These theorems are discussed and applied both with and without a remainder. To show the necessity for the vanishing of R_n, the function $f(x) = e^{-1/x^2}$ is expanded by Maclaurin's series, and as e^{-1/x^2} and all its derivatives vanish for $x = 0$ the expansion is the same as the expansion of $f(x)$ and therefore fails. Nothing impresses upon the mind of the student the necessity for $R_n \doteq 0$ so much as to see such an example. Throughout the work the conditions such as the above, which are apt to impress the student as being made to be on the side of safety, are shown to be necessary by means of a well chosen example.

Book IV. Complex numbers and quaternions. — In the discussion of imaginaries the author makes the greatest departure from the general method pursued. Imaginaries are defined by the Argand diagram, which, though elegant, seems to be rather out of keeping with the analytic development of the remainder of the book. A short discussion of functions of a complex variable is also given, but as no further mention is made of the subject it is sufficient. Quaternions are defined and discussed as an extension of imaginaries.

Book V. Algebraic equations. — The book is divided into two main parts: existence and number of roots, and solution of equations. Elimination is treated before the existence of a root is proved, and Euler's method of elimination is given without the assumption of a root. An attractive though short treatment of symmetric and multiple-valued functions of the roots is given. The invariants of the binary form are given as functions of the roots. In giving the methods of approximate solutions it is noticeable that Horner's method is omitted.

Book VI. Differential calculus. — The term differential calculus is used in the sense that differentials and not derivatives are used. In fact different words are used for the operations of finding differential and derivative. The book opens with a clear discussion of differentials in which various assumptions as the nature of differential are made and discussed. Change of variable is first discussed by means of differentials. In the early part of the book one is made to see clearly both the advantages and disadvantages of using differentials. Derivation in a given direction, so useful to the physicist, is discussed briefly but to the point. The Jacobian, Hessian, and first and second differential parameters are also treated at some length. The remainder of the book is taken up with the applications of the differential calculus to geometry, including the theory of twisted curves and the differential geometry of surfaces. In the differential geometry ordinary x, y, z coördinates are used, just a slight mention being made of gaussian coördinates. The care with which definitions are given deserves special notice: for example, an envelope is defined as the limit of the point of intersection of consecutive curves of the family $f(x, y, \alpha) = 0$, or if the curves do not intersect as the locus of the points whose distance apart is an infinitesimal of higher order. This last statement brings under the definition the envelope of such families as $y = (x - a)^3$, which is not included by the definition given by Salmon and others.

Book VII. Integral calculus. — In the applications of the differential calculus a few problems which required the finding of the antiderivative were solved; but here, after mere mention of integration as the inverse of differentiation, it is defined as a summation. The conditions for the existence of

$$\int_a^b f(x)dx,$$

the conditions for the integrability of a series, etc., are clearly and elegantly set forth. The various methods of integration are discussed quite fully, especially integration by substitution, but unfortunately no good example is given of the absurd results obtained by an unwise use of the method. The principal applications made of the integral calculus are finding lengths and areas of plane curves and area and volume of surfaces. The book closes with a short but fairly complete chapter on differential equations, treating especially those types which are integrable, but such ones as the Riccati equation are discussed at some length.

The appendix contains chapters on the Weierstrass function; calculus of differences, developed far enough to obtain the formula for interpolation; properties of Bernoulli's numbers, which have been previously defined as numbers satisfying the equation

$$(B + 1)^p - (B)^p = p.$$

The calculus of variations is developed far enough to obtain the equation of geodesic lines.

In the preceding pages I have endeavored to show the value of the work to the beginner, but this is by no means its only good quality; however space forbids touching on all. On account of the clear and scientific presentation and the numerous, well chosen illustrative examples, I know of no book which, placed in the hand of the beginner, would lead him more surely to a proper appreciation of the infinitesimal calculus.

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SHORTER NOTICES.

Grandeurs Géométriques. J. PIONCHON. Grenoble, A. Gratier et J. Rey. Paris, Gauthier-Villars, 1903. 128 pp.

THIS little volume is one of a series of seventy, constituting the Bibliothèque de l'élève ingénieur of which M. Pionchon is the general editor. This library consists of five sections: mathematics, mechanics, industrial physics, industrial electricity, and industrial economics. Each volume of about 150 pages is to contain an exposition of the fundamental notions, from the theoretical as well as practical point of view, of the subject with which it deals, and is intended to serve as a basis for later more detailed study. This plan we believe is novel and commendable. The serviceableness of the collection will of course depend largely on the way in which the plan is carried out. Judging by the volume before us, we should say that the little library promises to fulfill in a very large measure the hopes of its creator. M. Pionchon has succeeded in presenting in attractive form and logical sequence the definitions and more important properties of the fundamental geometric concepts, and the methods for the evaluation of various geometric quantities.

The author has confined himself to the mere statement of results whenever the proof is long, but is careful to show the interdependence of theorems whenever the relations are simple. He succeeds by this means in keeping alive the interest of the reader, who would soon tire of a mere list of properties and formulas. The ground covered is remarkable considering the elementary character of the treatment and the small amount of space used. We find a treatment, *e. g.*, of the notions of principal normal, binormal, curvature, torsion, osculating circle, osculating helix, in the case of twisted curves; and in the case of surfaces we find the definitions and more important properties of principal radii of curvature, mean and total curvature, ruled, developable, and minimal surfaces, lines of curvature, etc., subjects, many of them, that an American student of engineering never hears anything about. The book also contains a large number of formulas, which are made easily accessible by means of a good index.

The treatment of some of the subjects could easily be criticised on the score of rigor and some of the propositions appear to be stated with too much generality; but the niceties of modern rigor must not be insisted upon in such an elementary and one might almost say popular exposition, and the inaccuracies we have noted may well be pardoned in view of the general excellence of the whole. The typographical errors that we have noticed are few, and all of such an evident character, that it is quite unnecessary to enumerate them. We believe that M. Pionchon is to be congratulated on writing a thoroughly serviceable and very readable book.

J. W. YOUNG.

Étude sur les Quantitées mathématiques. Grandeurs dirigées. Quaternions. CLARO CORNELIO DASSEN. Paris, A. Hermann, 1903. vi + 133 pp.

M. DASSEN tells us in the introduction that it has been his object in writing this book, to "clear up and popularize the notions which lie at the foundations of pure mathematics." The author takes the extreme utilitarian view, and will admit into the science no "play of definitions and symbols" which cannot be put to "some practical use," lest he be beguiled into a realm of mere cabalistic hocus-pocus. He admits, however, that some intrinsically useless investigations may have a certain indirect value, and then gives us a hint on the breadth of his mathematical learning as follows: "The so-called non-euclidean geometry, for example, though useless in itself, because it does not correspond to experience, has nevertheless shown itself indirectly useful in proving that the euclidean geometry, the only one that does correspond to experience, is not apodictically true and has hence served to refute the arguments of Kant on the a-priority of the concept of space" (page 2).

With these introductory sentiments the author takes up his work of clearing up the notions that form the basis of algebra. A mathematical quantity is defined (page 3) as "anything that can be thought of as consisting of an aggregate of parts or 'units' such that they may be subjected to combinations called 'operations' either among themselves or with analogous elements of other aggregates having the same unit." Subtraction (page 14) is pronounced impossible when the minuend is less than the subtrahend, and (page 17) the symbol $-a$ is declared meaningless and incapable of interpretation, unless to the notion of quantity be added that of direction (the discussion of which is reserved for the later chapters). Yet a few pages further on (page 20), we are told that "in order to preserve the rules of calculation," it is necessary to give a certain meaning to the symbols a^0 and a^{-n}, and are referred to the great advantage resulting from the use of this notation, with never a word in justification of operating with the "meaningless" symbol $-n$.

We agreed with the author (page 17), when he showed that the symbol $0/0$ could represent any quantity; but find some difficulty in realizing (page 20) that "0^0 evidently signifies zero," when the symbol a^0 has just been defined by the relation $a^m/a^n = a^{m-n}$. But the author certainly means what he says, for later (page 25) we find: "The expression $\log_0 0$ equals zero (for $0^0 = 0$)." It seems to the reviewer that the author in pursuance of his avowed object should have elucidated this point a little more fully.

The author probably reaches the highest point in his progress toward the simplification of mathematics in his proof of the existence of $\sqrt[b]{a}$. He observes (page 26) that all continuous mathematical quantities can be represented by lengths, and then says: "Now it is easy, by graphical constructions, to determine lengths connected with others by the relation $c^b = a$, where b is any whole number and a any length whatever, whence $\cdots c = \sqrt[b]{a}$, a result which shows that even when $\sqrt[b]{a}$ is meaningless in itself it corresponds nevertheless to a length." He then describes "for example" the necessary construction for $b = 2$. We wonder if M. Dassen could not have given a more complete treatment of this point. We are not in the habit of regarding constructions requiring complicated linkages or recourse to hyperspace as "easy." That the author regards the above as a valid proof is clear from the fact that later (page 28) he speaks of "the quantity $\sqrt[b]{a}$, of which we have above shown the existence." We are told further (page 28) that in case of continuous magnitude the presence of quantities other than the rational is necessarily imposed by the assumption of the infinite divisibility of such magnitude, "for otherwise the division into parts would be necessarily limited." This sounds very clear and simple; and yet we are troubled by the fact that infinite divisibility in M. Dassen's sense seems to us to imply not necessarily a continuum, but only an aggregate everywhere dense in itself, for the representation of which the rational numbers suffice.

Enough has probably been said to indicate the character of M. Dassen's philosophy and the mathematical knowledge on which it is based, so that we may refer anyone interested in its further development to the remaining two chapters of the book itself, which have to do with directed quantities in the plane and in space respectively. In an appended note the author pays his respects to the work of Tannery and Kronecker regarding the founding of analysis on the concept of the positive integer alone, and pronounces it quite useless and a mere jugglery of symbols, "at which one is justly shocked." From what precedes, the fact that he has entirely missed the real object of such work is not surprising.

Before closing, we would, however, refer to a feature of the work which is of considerable interest. The author has scattered through the text a very large number of historical data. These are quite independent of his philosophy and seem to be drawn from reliable and often not easily accessible sources.

J. W. YOUNG.

A Treatise on Differential Equations. By ANDREW RUSSELL FORSYTH. Third Edition. Macmillan and Co., 1903.

IT is not an easy matter to review a book which, like the present one, has been before the public so many years, the first edition having appeared in 1885 and the second in 1888. That this treatise has many virtues has been quite conclusively shown by its success. In fact in English-speaking countries the domain of differential equations has, since 1885, been synonymous with the name of Mr. Forsyth, at least in the minds of that great category of students whose knowledge comes from text-books only.

As to the arrangement and the general contents of the book it would be useless to make any comment; it is all so well known. The book is a most useful one. It is very much to be regretted however that, in a considerable number of instances, the statements are given in a loose and misleading manner. On page 4 for example we find the following remark: "In the example considered, the equation giving dy/dx had only a single root; when it is of the form

$$\left(\frac{dy}{dx}\right)^2 + P\frac{dy}{dx} + Q = 0,$$

then the integral equation will be of the form

$$A^2 + AP' + Q' = 0,$$

where A is an arbitrary constant, etc."

Clearly, this statement without any restriction as to the domain of rationality is without value, and in a book intended for beginners it is misleading. Similarly the statement of the problem of integrating a differential equation given on page 6 is destined to perpetuate an erroneous conception. The author says in italics: "In fact every differential equation is considered as solved, when the value of the dependent variable is expressed as a function of the independent variable by means either of known functions or of integrals, whether the integration in the latter can or cannot be expressed in terms of functions already known." Again on page 51 he says: "Cases occur in which reduction to quadratures is not possible, that is to say, *the equation cannot be solved analytically*." (The italics are mine.) Of course the author knows that analytic integration of equations not reducible to quadratures *is* possible; his great work on the "Theory of differential equations," which was reviewed in this BULLETIN some years ago, shows him to be thoroughly familiar with the modern ideas. But no one would imagine this to be the case from reading his "Treatise." Of course it is justifiable to divide the theory of differential equations into two parts, one accessible by elementary means and the other depending upon the most advanced ideas of modern analysis. But why make statements in the elementary portion which must be contradicted later on, when a few words would suffice to explain the situation?

Lie's theory of continuous groups plays such an important part in the elementary problems of the theory of differential equations, and has proved to be such a powerful weapon in the hands of competent mathematicians, that a work on differential equations, of even the most modest scope, appears decidedly incomplete without some account of it. It is to be regretted that Mr. Forsyth has not introduced this theory into his treatise. The introduction of a brief account of Runge's method for numerical integration is a very valuable addition to the third edition. The treatment of the Riccati equation has been modified. The theorem that the cross ratio of any four solutions is constant is demonstrated but not explicitly enunciated, which is much to be regretted. From the point of view of the geometric applications, this is the most important property of the equations of the Riccati type. The theory of total differential equations has been discussed more fully than before, and the treatment on the whole is lucid. The same may be said of the modified treatment adopted by the author for the linear partial differential equations of the first order. A very valuable feature of the book is the list of examples.

E. J. WILCZYNSKI.

Introduction to Projective Geometry and Its Applications. By ARNOLD EMCH. New York, John Wiley & Sons, 1905. vii + 267 pp.

To some persons the term projective geometry has come to stand only for that pure science of non-metric relations in space which was founded by von Staudt. To others the original significance of the word, implying an actual projection of one metrical space upon another, still remains essential. The author of this book belongs to the latter class. He starts from metric and descriptive geometry. In the development of the matter treated in the text there is no trace of any kind of purism. Analytic and synthetic methods are everywhere used impartially. The result is a book which will certainly appeal to students of engineering and others who desire to use projective geometry in practical work.

Although there is to be found, especially in the later chapters, much which should interest students of pure mathematics, there are a number of defects which cannot but detract from their enjoyment of the work. These seem to be in a great measure a matter of style. Thus on page 19 we find in italics :

"An involution of rays contains one, but only one, rectangular pair." On the next page, likewise in italics, is a second theorem: "An involution having more than one rectangular pair has all its pairs rectangular." These two theorems are, on their face, somewhat contradictory. Of course there is no difficulty in understanding their relation to each other, and there might be little cause for comment were it not for the fact that too many instances of this sort of infelicity in style are present. On page 68 the author states: "A plane figure may therefore be considered either as a configuration of points or as a configuration of straight lines. This is the principle of duality." Such a statement seems hardly definite enough to cover the case. The matter of dividing by factors which may be zero, the various special cases which may arise, in fact the whole modern demand for a greater accuracy in geometric work is not sufficiently regarded.* Apart from this the work has much to commend it. And those for whom the book was especially written will probably not think the defect very serious. To the reader, whoever he be, the uncommonly good index will be of great service.

E. B. WILSON.

Differential- und Integralrechnung; Zweiter Band: Integralrechnung. Von W. FRANZ MEYER. Mit 36 Figuren. Leipzig, C. J. Göschensche Verlagshandlung (Sammlung Schubert, number XI), 1905. 16 + 443 pp.

THE present volume is a direct continuation of the preceding one on differential calculus, to which constant references are made. Integration is first defined as the inverse of differentiation. It is treated analytically and applied to elementary algebraic, logarithmic, and trigonometric functions. The idea of summation is introduced by a very detailed discussion of successive approximations to the area of the parabola, then extended to any plane curve, first with equal intervals, then for any law of division. The oscillation of a function, and superior and inferior integrals are repeatedly mentioned and strongly emphasized.

The first theorem of mean value is introduced by the aid of a figure, then made precise and applied to several problems.

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* See for example M. Bôcher on "A problem in analytic geometry with a moral," *Annals of Math.*, vol. 7, p. 44 (October, 1905).

Areas defined by polar coördinates are first treated by means of transformation of the variables after the rectangular coördinates have been discussed; but a good geometric discussion is added. Maclaurin's series and allied formulas are treated at some length, the forms of the remainder being minutely compared with the forms derived in differential calculus, but the difficulties of a double or multiple limit receive but little attention.

Chapter 2 is concerned with quadrature, complanation, and rectification. The idea of length is carefully defined and a very large number of illustrations are brought in, but all are worked out in full, leaving no problems for the student. The two following chapters are devoted to tangents and normals, contact and curvature, and inflexions, treated by the usual methods of the differential calculus. The author explains in the preface that these subjects were excluded from the preceding volume on account of its size.

It seems remarkable that when nearly three hundred pages have been covered, over half of them dealing with applications, only a very limited number of forms of integrals have been derived. Much space could have been spared by a different arrangement. In the latter part of the book (pages 281–370) a long chapter is devoted to systematic integration, very similar to the plan followed by the better American text-books, though many more details are given, leaving fewer steps for the student to supply. As a natural extension of this chapter another follows on elliptic integrals, the addition theorems of elliptic, hyperbolic, and circular functions, and integration in series. The last topic, double integration, occupies about twenty pages. These include the fundamental definitions, discussion of limits, change of sequence, transformations of the variables, and a few applications to cubature and complanation.

The book contains a number of instructive features, many problems are treated by new methods, but the arrangement of topics seems unfortunate for a student. The book is full of helpful cross references, but is not provided with an index.

VIRGIL SNYDER.

Le Calcul des Résidus et ses Applications à la Théorie des Fonctions. By ERNST LINDELÖF. Paris, Gauthier-Villars, 1905. 141 pp.

WE here have a small volume devoted to a relatively narrow though important field of mathematics. The calculus of residues, as used in the sense of the author, comprises such results as flow in a direct manner from Cauchy's integral theorem for functions of a complex variable — *i. e.* from the theorem "the integral of a function $f(z)$ of complex variable z about a closed contour upon which this function is single-valued and analytic and within which the same function is single-valued and analytic except for the poles $z = \lambda_1, \lambda_2, \cdots, \lambda_n$ is equal to the sum of the residues of these poles." A familiar example of such results is furnished by the application so frequently made of the theorem for the evaluation of definite integrals, an application due to Cauchy himself and highly developed by him. Likewise, it has long been known that the theorem may be made to yield valuable results in the study of the convergence of Fourier's series. Within recent years Cauchy's investigations in this field have been worked over and many additions have been made to the list of applications of the theorem, which list now includes convergence proofs for several of the fundamental series developments of mathematical physics and important additions to our knowledge of the properties of functions defined by power series. As the field has widened and its importance has become more and more recognized, the need of some systematic exposition of it has naturally been created and we may therefore at once extend to this pioneer volume of Professor Lindelöf a hearty welcome.

Chapter I (Principes et théorèmes fondamentaux, pages 1–20) contains such material from the general theory of functions as is needed for reference in the subsequent pages. It seems somewhat surprising, in view of the special character of the book, that the author should take the trouble here to give proofs. For example, it appears superfluous in such a work to *prove* Laurent's theorem. A mere tabulation of the needed fundamental theorems and definitions would apparently suffice.

In Chapter II (Applications diverses du calcul des résidus, pages 20–52) we soon see how Cauchy's integral theorem yields a wide variety of interesting relations, some old and some new. For example, it thus appears that for the kth Bernoulli number B_k we may write

$$B_k = \frac{2(2k)!}{(2\pi)^{2k}} \sum_1^\infty \frac{1}{\nu^{2k}}.$$

The chapter closes with nine pages devoted to the study of definite integrals (after the celebrated method of Cauchy) from the standpoint of the calculus of residues. By means of well-chosen examples it is seen how powerful the method is, and the reader is thus left with a clear impression of the assistance derived from the theory of functions of a complex variable in the study of functions of a real variable.

In Chapter III (Formules sommatoires tirées du calcul des résidus, pages 52–87) the main object of discussion is the infinite series

$$\sum_{m}^{\infty} f(\nu),$$

$f(\nu)$ being a function of ν subject to certain preliminary conditions. When these conditions are satisfied it is shown that the calculus of residues enables one to express the sum of the series by means of a finite number of definite (improper) integrals. For example, we may thus write

$$\sum_{m}^{\infty} f(\nu) = \int_{a}^{\infty} f(\tau)d\tau + \int_{a}^{a+i\infty} \frac{f(z)dz}{e^{-2\pi iz}-1} + \int_{a}^{a-i\infty} \frac{f(z)dz}{e^{2\pi iz}-1}$$
$$(m-1 < a < m).$$

The utility of this formula is at once apparent, for if we can sum a series in this form we are thereby in possession of something from which the properties of the series (in general difficult to obtain) may be deduced. It is in precisely this way that the study of infinite series has been notably enriched within recent years through the calculus of residues. The author, having once established these "formules sommatoires," virtually devotes the remainder of his volume to pointing out their important applications. First, a variety of results due to Euler, Maclaurin, Hermite, Darboux and others are obtained as special consequences, then the bearing of the same formulas upon the study of the convergence of Fourier's series (a bearing well known to Cauchy himself) is briefly pointed out.

Two particular points are perhaps worthy of note before dismissing the chapter. On page 57 where several conditions are imposed upon $f(z)$ the author requires in the first of the conditions numbered 2° that the equality

$$\lim_{t=\pm\infty} e^{-2\pi t} f(\tau + it) = 0$$

shall hold true *uniformly* for $a \leqq \tau \leqq \beta$. We would note that the equality signs are here superfluous — *i. e.*, the condition upon τ may be written simply $\alpha < \tau < \beta$. (Cf. Stolz, Allgemeine Arithmetik, volume I (1885), page 273.) Likewise, the second of the conditions numbered 2° would be better if it read simply "La condition (A) est vérifiée uniformément pour $\tau > a$." Similar remarks may be made at numerous places throughout the work where the notion of uniform convergence is introduced.

Secondly, a fruitful source of investigation is suggested on page 64, where the author calls attention to the alteration required in the formulas just preceding when it is assumed that $f(z)$ possesses a finite number of singular points in the region specified at the top of the page. Instead of the number of such points being finite, suppose a case where they are infinite. Then the required alteration introduces an infinite series of residues into the formula in question. Can we still express

$$\sum_{m}^{\infty} f(\nu)$$

by means of a finite number of definite integrals; if so, when?

Chapter IV (Les fonctions $\Gamma(x)$, $\zeta(s)$, $\zeta(s, w)$, pages 87–108) is devoted principally to the discussion by means of the formulas of chapter III of the function $\log \Gamma(x)$ and its derivatives and of the socalled function of Riemann $\zeta(s)$.

In Chapter V (Applications au prolongement analytique et à l'étude asymptotique des fonctions définies par un développement de Taylor, pages 108–141) we see clearly the bearing of the calculus of residues upon the problem of analytic extension for functions defined by power series: *i. e.*, the problem of determining the value of the function $F(x)$ at any point in its domain of existence when this function is defined by the series

$$F(x) = \phi(0) + \phi(1)x + \phi(2)x^2 + \cdots + \phi(\nu)x^\nu + \cdots$$

(*radius of convergence finite and different from zero*).

Under specified conditions for $\phi(\nu)$ it is found that the function $F(x)$ exists and is analytic at all points x except those which lie within a certain segment of the plane containing the real axis of x, and an explicit form for this function in terms of definite integrals is obtained. The chapter for the most part is an exposition of the author's own investigations upon the subject, as contained in his memoir "Quelques applications d'une formule sommatoire générale" (*Acta Societatis Scientiarum Fennicae*, volume 31, number 3 (1902)). It must be admitted that this chapter is difficult reading, a feature which seems due in large measure to the involved character of the hypotheses placed upon $\phi(\nu)$. In this connection may we not question which course would here be preferable, to employ involved hypotheses like these and thus attain a degree of generality in the results as great as that obtained by all previous investigators, or to sacrifice generality in some measure for the sake of simplicity and attractiveness? In a treatise like this the reviewer believes the latter alternative preferable.

Finally, we will venture to make one general remark. Professor Lindelöf's work being confined merely to the applications of the calculus of residues to the theory of functions as such, it is not surprising that nowhere do we find mention of the profound application of this calculus which Dini has made in the study of the convergence of the important series developments of mathematical physics. As early as 1880 his work "Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale" appeared, containing rigorous convergence proofs based upon the calculus of residues for Fourier's series, series in terms of Bessel's functions, zonal harmonics and elliptic functions. Attention is especially directed to the portion of this work from page 139 to page 328. Any general treatise bearing the title used by our author should certainly dwell more at length upon this latter aspect of the subject. Let us therefore hope to see in the near future a similar treatise having a wider scope.

WALTER B. FORD.

Geschichte der Elementar-Mathematik in systematischer Darstellung. By Dr. JOHANNES TROPFKE, Oberlehrer am Friedrich-Realgymnasium zu Berlin. Bd. I. Rechnen und Algebra, Leipzig, 1902. viii + 332 pp. Bd. II. Geometrie, Logarithmen, Sphärik und Sphärische Trigonometrie, Reihen, Zinseszinsrechnung, Kombinatorik und Wahrscheinlichkeitsrechnung, Kettenbrüche, Stereometrie, Analytische Geometrie, Kegelschnitte, Maxima und Mimima. Leipzig, 1903. viii + 496 pp.

THIS work of 844 large pages is a most welcome addition to the few histories of elementary mathematics now easily accessible. For all but those actively engaged in historical research, Cantor's Geschichte der Mathematik has become the first refuge when in quest of information on points relative to the history of mathematics. But the use of Cantor's work is hampered by two grave restrictions; it stops short a century and a half before the present time, and its order of development is chronological, grouping together the work be it of a man, or of a school or of a period of time. This makes it quite laborious to get a connected view of the history of any subject or topic. But it is precisely this which any teacher must have who wishes to utilize the history of mathematics to lend additional interest to his instruction. Therefore, the best thing for one to do who wished a more extensive view than that of brief works like Fink's Geschichte der Elementar-Mathematik* has been to collect fragments from Cantor by following the clues given in the index, and weave them together for himself as best he might into a more or less coherent history of the development of the subject or topic in hand. This has been done, no doubt, by many a one on both sides of the Atlantic, among them our author, who in the course of years has gathered together a huge mass of material, in the first instance from Cantor, and then from other works on the history of mathematics and from journals, these in turn leading him to the original sources. The extent and thoroughness of the study of the sources is evidenced by the number and exactitude of the references in the footnotes. Of them there are 1233 in the first volume and 1836 in the second, a total of 3069. Every statement of consequence in the text is supported by such a reference either to the original sources, or to Cantor or other works on the history of mathematics, or to both. These references are always very specific and furnish most precious clews to those who wish to trace back historical statements to their ultimate sources. An excellent mode of beginning the careful study of the history of any particular topic would be to look up in the originals the references given by Tropfke. These in turn would in many instances suggest others and the student would soon find himself launched on a research of his own. We need to pay more attention to the history of mathematics in our universities and colleges, not only to the interesting and instructive material already accumulated, but to the active enrichment of the fund.

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* Translation by Beman and Smith, Open Court Publishing Co., Chicago.

In such undertakings, so far as related to elementary mathematics, Tropfke's work will prove a most valuable and suggestive help.

The work will prove no less helpful to the teacher of elementary mathematics. Every teacher recognizes the desirability of attention to the history of his subject, both as giving more complete understanding of it, and as enhancing interest; but, as already mentioned, the serious labor of collecting the data confronts him. Several excellent and well-known works, usually arranged chronologically, already exist, which can be drawn on for this purpose; the work under review, however, is larger, exceptionally well arranged, and, being restricted to a narrow field, covers that field more thoroughly. Its style is readable, though condensed; the typography is fairly clear, and some attempt has been made to aid the eye by the use of different styles of type. A somewhat more open page would be acceptable, but would, perhaps, increase unduly the bulk of a work which is already quite large.

The history of arithmetic is first taken up and occupies 120 pages of the first volume. In algebra, algebraic symbols, the number concept, algebraic operations, proportions, equations to the fourth degree inclusive are taken up, and a few pages are devoted to equations of degree higher than the fourth and diophantine equations.

The detailed title of the second volume given above indicates its scope sufficiently. The work includes the conic sections and the methods of analytic geometry, but does not include the calculus; its field is therefore substantially mathematics as taught from the beginning of the work in the grades to the close of the freshman year in college.

It would not be difficult to find instances in which different treatment might be advocated (the modern definitions of irrational number, for example, deserve more than the bare mention which they receive); and in such an enormous mass of facts and references, misprints and errors of various sorts are surely to be found for the seeking, but in view of the high excellence of the whole work it would be ungracious to lay stress on minor defects. Whether regarded from the point of view of thoroughness of treatment, of convenience of arrangement, of conscientiousness in citing authorities or of scholarly spirit, the work as a whole deserves only high praise and ranks as a most valuable contribution to the history of elementary mathematics.

J. W. A. YOUNG.

Propagation de l'Electricité: Histoire et Théorie. By MARCEL BRILLOUIN. A. Hermann, Paris, 1904. ii + 398 pp.

THIS valuable work, as the author says in his preface, is "la reproduction assez fidèle des leçons que j'ai professées au Collège de France pendant l'année 1901–1902. Ce n'est nullement un traité d'électricité complet et méthodique; c'est, conformément, je crois, à l'esprit de l'enseignement du Collège de France, un ensemble de leçons, très inégalement développées, suivant que le sujet dont elles traitent est plus ou moins connu par des publications françaises, ou qu'il m'a paru comporter quelques remarques historiques ou théoriques nouvelles."

The author expects to follow this volume by another, containing the essential portions of his lectures for 1902–1903 and 1903–1904 on electron theory.

The first of the four books which make up the present volume is devoted to the pioneers of the science from Cavendish to Kirchhoff and Clausius. Cavendish and Ohm receive separate chapters, which are biographical as well as scientific, and particularly interesting. In this book, as in most other portions of the work, numerous references are given.

The second book treats of steady currents, and also of changing currents without magnetic induction. The first chapter, entitled "currents in space," considers, among other matters, the decay with time of electrification in a conducting medium, electric double layers, and the methods of Gouy and Cohn and Arons for the determination of the dielectric constants of conducting liquids. In the treatment of the first mentioned subject the relaxation time, involving the dielectric constant, is given for a number of substances, including bismuth and copper; but the dielectric constants of these substances are not given, nor is any authority quoted, while the well known dielectric constant of water is given — a procedure which can hardly be justified in view of the little that is known of the dielectric constants of metals. The second and third chapters are devoted to the resistances of conductors with electrodes of relatively high conductivity and to Rayleigh's method of approximate resistance evaluation. The resistance of a circular cylinder, with various electrodes, is treated at length by Bessel's functions, to some of the properties of which a separate chapter is devoted. Electric propagation along a cable is discussed in a long chapter after Kelvin, Kirchhoff, and Vaschy; and the book ends with two chapters on the much neglected subject of the electric field of the steady current. Attention should be called to the erroneous statement at the beginning of the book to the effect that all following equations are written in electromagnetic units. As a matter of fact the author continually expresses K, the dielectric constant, in the electrostatic unit, and therefore introduces the square of Ω, the ratio of the electromagnetic unit charge to the electrostatic unit charge, to make his equations correct. Unfortunately, M. Brillouin is not alone in this practice.

The third book, on electromagnetic induction, is introduced by a brief but excellent historical chapter. Joseph Henry, however, is not mentioned, and Faraday receives scarcely better treatment. Four chapters are devoted to induction in fixed circuits; special attention being given to parallel wires, cylindrical coils, and *spherical* coils. On account of the simplicity of the exact formulæ and the facility of construction, the author recommends the latter form of coil for inductance standards. A chapter is devoted to the diffusion of currents in conductors, and the book ends with two chapters on electric propagation along conductors devoted largely to Kirchhoff's classic memoir of 1857.

The fourth and concluding book treats of the general electromagnetic field. After a discussion of the more general equations, involving a comparison of the theory of Maxwell with the older theories of Neumann and Helmholtz, the field of the Hertzian oscillator is treated at length, first without damping, after Hertz, and then with damping, after Pearson and Lee. The two final chapters of the work are devoted to the electric oscillations of a sphere and the electric oscillations of a prolate spheroid. Extensive numerical tables for the latter are given at the end of the volume.

The treatment appears to us to be, in general, accurate, elegant, and commendably concise. Too great brevity, however, has in some places resulted in obscurity. The book contains valuable critical remarks and considerable original work by the author not hitherto published, as well as other valuable matter not readily accessible elsewhere. The errata we have noticed are not numerous and are for the most part typographical and not likely to be misleading. Confusion will doubtless be produced in the minds of some readers by the author's indiscriminate use of $\triangle$ for both $\triangle$ and ∇ as ordinarily employed.

The work naturally contains little to interest the student of pure mathematics, but it can be highly recommended to the physicist and forms an acceptable addition to electrical literature.

S. J. BARNETT.

NOTES.

THE annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held on Thursday and Friday, December 28–29. The Council will meet on Thursday morning, and the annual election of officers and other members of the Council will close on Friday morning. The usual informal dinner will be arranged for Thursday evening.

THE Chicago Section will hold its eighteenth regular meeting at the University of Chicago, on December 29–30. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section, Professor THOMAS F. HOLGATE, 617 Library Street, Evanston, Ill., not later than December 5.

THE concluding (October) number of volume 6 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Sur l'écart de deux courbes et sur les courbes limites," by M. FRÉCHET; "On a certain system of conjugate lines on a surface connected with Euler's transformation," by J. EIESLAND; "Surfaces of constant curvature and their transformations," by L. P. EISENHART; "Volumes and areas," by N. J. LENNES; "On a problem including that of several bodies and admitting of an additional integral," by E. O. LOVETT; "On the stability of the motion of a viscous liquid," by F. R. SHARPE; "Ueber die vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören," by A. LOEWY; "On the Cayley-Veronese class of configurations," by W. B. CARVER. Also, Notes and Errata, volumes 5–6; and table of contents, volume 6.

THE October number (volume 7, number 1) of the *Annals of Mathematics* contains: "Concerning Green's theorem and the Cauchy-Riemann differential equations," by M. B. PORTER; "On the singularities of tortuous curves," by P. SAUREL; "On the twist of a tortuous curve," by P. SAUREL; "The continuum as a type of order: an exposition of the modern theory, V–VI. With an appendix on the transfinite numbers," by E. V. HUNTINGTON; "A problem in analytic geometry with a moral," by M. BÔCHER.

THE concluding (October) number of volume 27 of the *American Journal of Mathematics* contains: "Concerning certain 4-space quintic configurations of point ranges and congruences, and the sphere analogues in ordinary space," by C. J. KEYSER; "Some relations between number theory and group theory," by G. A. MILLER; "The differential invariants of space," by J. E. WRIGHT; "An arithmetic treatment of some problems in analysis situs," by L. D. AMES; "On the definition of reducible hypercomplex number systems, II," by H. B. LEONARD.

THE last number (double number 8 and 9, volume 14) of the *Jahresbericht der deutschen Mathematiker-Vereinigung* contains a good portrait of Sir W. R. Hamilton, together with an appreciative estimate of his scientific work by Professor E. STUDY. The address of the editor, Professor Dr. A. GUTZMER, is now Halle a. S., Martinsberg 8 I.

THE mathematical section of the California teachers association will hold its first regular meeting at the University of California on December 26. Professors I. STRINGHAM and J. B. CLARKE will read papers before the section.

THE publishing house of B. G. Teubner, in Leipzig, announces that the following books are in press: H. BRUNS, Wahrscheinlichkeitsrechnung und Kollektivmasslehre; N. NIELSEN, Handbuch der Theorie der Gammafunktionen; FR. ROGEL, Das Rechnen mit Vorteil; E. J. WILCZYNSKI, Projective differential geometry of curves and ruled surfaces (in English).

THE technical schools named below offer the following courses in mathematics for the present winter semester:

AACHEN. — By Professor F. JÜRGENS: Higher mathematics, I. — By Professor F. KÖTTER: Descriptive geometry; Graphical statics. — By ——— (Successor to Professor L. Heffter): Higher mathematics, II, with exercises; Seminar. — By Professor A. SOMMERFELD: Mechanics, I and II.

BERLIN. — By Professor O. DZIOBEK: Analytic geometry Differential and integral calculus. — By Professor E. HAENTZ-

SCHEL: Analytic geometry; Differential and integral calculus. — By ——— : Descriptive geometry, I. — By Professor H. HERZEN: Descriptive geometry, I. — By Professor G. HETTNER: Analytic geometry; Differential and integral calculus; Theory of surfaces and twisted curves. — By Professor S. ZOLLES: Descriptive geometry, I; Graphical statics. — By Professor E. LAMPE: Analytic geometry; Differential and integral calculus; Definite integrals and differential equations. — By Professor E. STEINITZ: Theory of potential; Theory of functions; Algebra. — By Professor G. HESSENBERG: Descriptive geometry, II. — By Professor S. KALISCHER; Elements of the theory of potential. — By Dr. R. MÜLLER: Differential and integral calculus.

BRÜNN. — By Professor E. WÄLSCH: Elementary mathematics, seven hours; Vector analysis, one hour. — By Professor O. BIERMANN: Selected chapters of higher mathematics, three hours; Approximation methods, two hours; Calculus of variations, one hour. — By Dr. E. FISCHER; Fourier's series, two hours; Theory of elimination, two hours; Exercises in higher mathematics, one hour. — By Professor O. RUPP: Descriptive geometry, six hours; with exercises, eight hours. — By Dr. F. OBENRAUCH: History of geometry, one hour.

BRUNSWICK. — By Professor R. DEDEKIND: Theory of numbers; Fourier's series. — By Professor R. FRICKE: Analytic geometry and algebra; Differential and integral calculus; Selected chapters of higher mathematics. — By Professor R. MÜLLER: Descriptive geometry with exercises; Theory of surfaces and twisted curves; Geometry of position.

DANZIG. — By Professor H. v. MANGOLDT: Higher mathematics, II. — By Professor F. SCHILLING: Descriptive geometry. — By Professor J. SOMMER: Higher mathematics, I.

DARMSTADT. — By Professor F. DINGELDEY: Higher mathematics, I and II. — By Professor P. FENNER: Trigonometry. — By Professor F. GRAEFE: Repertorium of higher mathematics; History of mathematics. — By Professor S. GUNDELFINGER: Higher mathematics, I. — By Professor L. HENNEBERG: Mechanics. — By Professor G. SCHEFFERS: Higher mathematics, I.; Descriptive geometry, I. — By Professor H. WIENER: Descriptive geometry, I and II. — By Dr. K. GAST: Practical geometry. — By Dr. W. SCHLINK: Selected chapters of analytic mechanics; Vector analysis.

DRESDEN. — By Professor M. KRAUSE: Differential calculus with exercises, five hours; Introduction to the theory of infinite processes, four hours; Seminar, one hour. — By Professor A. FUHRMANN: Applications of the calculus, two hours; Theory of surveying, two hours; Geodetic drawing, two hours. — By Professor G. HELM: Analytic geometry, II., four hours; Analytic mechanics, two hours. — By Professor M. DISTELI: Descriptive geometry, three hours; with exercises, four hours; Theory of surfaces and twisted curves, two hours. — By Professor E. NAETSCH: Selected chapters of the theory of partial differential equations, two hours.

GRAZ. — By Professor F. HOČEVAR: Algebra and analytic geometry, six hours; with exercises, two hours; Spherical trigonometry, one hour. — By Professor L. PEITHNER: Applications of the calculus to surfaces and twisted curves, four hours; with exercises, two hours. — By Professor K. STELZEL: Higher mathematics, four hours. — By Professor R. SCHÜSSLER: Descriptive geometry, four hours; with exercises, six hours; Theory of conics, three hours.

HANOVER. — By Professor L. KIEPERT: Differential and integral calculus, I, six hours; Geometry of position, three hours; Differential and integral calculus, III, two hours; Theory of differential equations and calculus of variations, two hours. — By Professor P. STÄCKEL: Differential and integral calculus, II, five hours; Elements of higher mathematics for architects and chemists, I, four hours; Analytic geometry of two and three dimensions, three hours. — By Professor C. RODENBERG: Descriptive geometry, I, nine hours; Descriptive geometry, II, nine hours. — By Dr. O. PETZOLD: Algebraic analysis and trigonometry, three hours; Exercises in the adjustment of errors, one hour.

KARLSRUHE. — By ——— (successor to Professor R. Haussner, who is now professor in Jena): Plane analytic geometry, three hours; Arithmetic and algebra, three hours; Trigonometry, three hours; Surfaces and twisted curves, two hours. — By Professor K. HEUN: Mechanics, I and II, six hours; Elementary mechanics, two hours. — By Professor A. KRAZER: Higher Mathematics, II, three hours; Exercises in hyperelliptic functions, two hours. — By Professor F. SCHUR: Descriptive geometry, four hours; with exercises, four hours;

Graphical statics, with exercises, four hours. — By Professor L. WEDEKIND: Higher mathematics, I, six hours; with exercises, two hours. — By Dr. G. HAMEL: Exercises in higher mathematics, one hour; Vector analysis with applications, two hours; Elements of mechanics, four hours.

MUNICH. — By Professor S. FINSTERWALDER: Higher mathematics, I, with exercises; Noneuclidean geometry. — By Professor A. v. BRAUNMÜHL: Elements of higher mathematics, III, with exercises; Seminar. — By Professor W. v. DYCK: Higher mathematics, I, with exercises; Theory of functions; Seminar. — By Professor L. BURMESTER: Descriptive geometry, I, with exercises.

PRAGUE (German). — By Professor K. ZSIGMONDY: Mathematics, I, six hours; Differential and integral calculus, two hours. — By Professor A. GRÜNWALD: Mathematics, II, five hours; Differential equations, two hours. — By Professor E. JANISCH: Descriptive geometry, four hours; with exercises, eight hours; Geometry of position, three hours. — By Dr. A. ADLER: Graphical calculation, two hours; Elements of geometry of motion, two hours.

STUTTGART. — By Professor R. MEHMKE: Descriptive geometry with exercises; Analytic mechanics with exercises; Seminar. — By Professor K. REUSCHLE: Discussion of curves; Analytic geometry of space with exercises; Invariants; Differential and integral calculus, II and III, with exercises; Seminar. — By Professor E. WÖLFFING: Elements of differential and integral calculus: Theory of functions, I. — By Dr. W. BRETSCHNEIDER: Review of elementary mathematics. — By Dr. J. STÜBLER: Elementary analysis.

THE Royal institute of Venice announces the following prize problem, for the solution of which it offers the Querini Stampalia prize of 3000 lire:

"To perfect in some important point the projective geometry of algebraic surfaces of two dimensions in space of n dimensions."

Competing memoirs should be written in Italian, French, English or German, and submitted to the secretary of the institute under the usual conditions before December 31, 1906.

PROFESSOR F. SEVERI, of Pisa, has been appointed professor of geometry at the University of Parma.

PROFESSOR C. SOMIGLIANA, of Pavia, has been appointed professor of mathematical physics at the University of Turin.

DR. H. BOURGET, master of mathematical conferences at the University of Toulouse, has been appointed professor of mathematics at the same institution.

DR. R. LEVAVASSEUR has been appointed master of mathematical conferences at the University of Lyons.

ON the occasion of the inauguration of President James of the University of Illinois, the honorary degree of doctor of laws was conferred upon Professor T. F. HOLGATE, of Northwestern University.

AT New York University, Dr. T. N. EDMONDSON, associate professor of physics, has been appointed professor of mathematics to succeed Professor P. LADUE, resigned. Dr. A. SCHULTZE, of the New York High School of Commerce, has been appointed assistant professor of mathematics.

DR. G. A. ARMSTRONG takes the place of Professor W. D. Cairns in the department of mathematics at Oberlin College during the latter's absence in Europe, and Mr. J. S. HAUCKEY has been appointed instructor in mathematics.

PROFESSOR ELEANOR DOAK has been made associate professor of mathematics at Mount Holyoke College, and granted a leave of absence for the present year. Miss CHRISTINE BUSBEE has been made instructor in mathematics in the same institution.

MR. C. R. MACINNIES and Mr. E. B. MORROW have been appointed instructors in mathematics at Princeton University.

MR. R. H. LEE has been appointed professor of mathematics at the Rhode Island State College.

MR. H. N. DAVIS has been appointed instructor in mathematics in Harvard University.

DR. J. V. WESTFALL has resigned the assistant professorship of mathematics in the State University of Iowa to accept an actuarial position in New York. Mr. W. E. BECK has resigned an instructorship in mathematics at the same university, having been appointed assistant in the computing division of the United States coast and geodetic survey. Mr. R. P. BAKER and Mr. C. M. THORNE have been appointed to fill the vacancies thus created.

MR. W. H. SHERK, of Oberlin College, has been appointed professor of mathematics at Buchtel College, Akron, Ohio.

MR. G. I. GAVETT, of Stanford University, has been appointed instructor in civil engineering at Cornell University.

MR. W. V. N. GARRETSON and Mr. C. E. LOVE have been appointed instructors in mathematics in the University of Michigan.

DR. H. B. LEONARD has been appointed instructor in mathematics at the University of Colorado.

DR. R. B. McCLENON, of Yale University, has been appointed instructor in mathematics at Iowa College, Grinnell, Iowa.

THE following have been appointed instructors in mathematics at the Pennsylvania State College: Mr. H. H. HIGLEY, of the East Stroudsburg Pennsylvania Normal School, Mr. G. A. WHITTEMORE, of Harvard University, and Mr. J. W. BEAL, of Colgate University.

MR. P. CAPRON, of Dummer Academy, has been appointed instructor in mathematics at Williams College, Williamstown, Mass.

MR. D. J. CRITTENBERGER and Mr. C. HASEMAN have been appointed assistants in mathematics at the University of Indiana.

MR. ARTHUR RANUM, formerly professor of mathematics in the University of Washington, has been appointed assistant in mathematics at Stanford University.

PROFESSOR FLORIAN CAJORI, of Colorado College, has received a half-year leave of absence, which he is spending at New York and Washington in collecting materials for his work as collaborator in extending Cantor's History of Mathematics through the period 1759–1799. Readers of the BULLETIN will be interested to learn that Professor Cajori has found in the New York Public Library (Lenox Branch) and in the

Columbia University Library copies of E. S. Bring's dissertation on the transformation of the quintic (Lund, 1786). Mr. Harley (*Quarterly Journal of Mathematics*, volume 6 (1863), page 43), could not learn of a complete copy outside of Sweden, except one at the Pulkova Observatory. Mr. C. Hill (*Öfersigt af Kongl. Vetenskaps-Acadamiens Förhandlinger*, 1861, page 317) speaks of its rarity even in Sweden.

DR. W. F. WISLICENUS, editor of the *Astronomisches Jahresbericht*, died October 3, at the age of 46 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ARRIGHI (G. L.). La storia della matematica in relazione con lo sviluppo del pensiero. Torino, Paravia, 1905. 16mo. 13 + 133 pp. L. 1.50

BÄHR (E.). Abbildung einer unendlichen Ebene, die durch Aufschlitzen längs zweier senkrecht zu einander stehenden Strecken zu einem zweifach zusammenhängenden Bereiche gemacht wird, auf ein Rechteck und einen Kreisring. (Diss.) Jena, 1905. 8vo. 40 pp.

BLYTHE (W. H.). On models of cubic surfaces. Cambridge, University Press (New York, Macmillan), 1905. 12mo. 12 + 106 pp. Cloth. $1.25

BOLYAI DE BOLYA (W.). Tentamen iuventutem studiosam in elementa matheseos purae elementaris ac sublimioris methodo intuitiva evidentiaque huic propria introducendi, cum appendice triplici. Editio secunda. Tomus II: Elementa geometriae et appendices. Mandato Academiae scientiarum hungaricae suis adnotationibus adiectis ediderunt J. Kürschák, M. Réthy, B. Tötössy de Zepethnek, Academiae scientiarum hungaricae sodales. Pars prima: Textus. 63 + 437 pp. Pars secunda: Figurae. 8 pp., 82 plates. Leipzig, Teubner, 1904. 4to.

BRICARD (R). Matematika terminaro kaj krestomatio. Paris, Hachette, 1905. 16mo. 61 pp. (Kolekto esperanta.)

CARSLAW (H. S.). Introduction to the infinitesimal calculus. Notes for the use of science and engineering students. London, Longmans, 1905. 8vo. Cloth. 5s.

DOSTOR (G.). Eléments de la théorie des déterminants, avec application à l'algèbre, la trigonométrie et la géométrie analytique dans le plan et dans l'espace, à l'usage des classes de mathématiques spéciales. 2e édition (nouveau tirage). Paris, Gauthier-Villars, 1905. 8vo. 33 + 362 pp. Fr. 8.00

FREY (H.). Ueber das Vorzeichen gewisser bestimmter Integrale. (Diss.) Heidelberg, 1905. 8vo. 47 pp.

GMEINER (J.). See STOLZ (O.).

GRAF (J. H.) Briefwechsel von Ludwig Schläfli mit Arthur Cayley. Mit dem Facsimile eines Briefes von A. Cayley. Festgabe der Universität Bern für das 50-jährige Jubiläum des eidgenössischen Polytechnikums in Zürich. 1855–1905. Bern, 1905. 8vo. 42 pp.

GUICHARD (C.). Sur les systèmes triplement indéterminés et sur les systèmes triple-orthogonaux. Paris, Gauthier-Villars, 1905. 12mo. 96 pp. (*Scientia*, No. 25.) Fr. 2.00

HEFFTER (L.) und KOEHLER (C.). Lehrbuch der analytischen Geometrie. (In 2 volumes.) Vol. I: Geometrie in den Grundgebilden erster Stufe und in der Ebene. Leipzig, Teubner, 1905. 8vo. 16 + 528 pp. M. 14.00

KISELJAK (M.). Grundlagen einer Zahlentheorie eines speziellen Systems von komplexen Grössen mit drei Einheiten. Bonn, 1905.

KOEHLER (C.). See HEFFTER (L.).

KOLLROS (L.). Un algorithme pour l'approximation simultanée de deux grandeurs. (Diss.) Zurich, 1905. 8vo. 48 pp.

KÜRSCHÁK (J.). See BOLYAI.

LÜBECK (O.). Differentialrechnung. 2te, neu bearbeitete Auflage. Strelitz, 1905. 8vo. 111 pp. M. 7.00

MELLOR (J. W.). Higher mathematics, for students of chemistry and physics, with special reference to practical work. 2d edition, enlarged. London and New York, Longmans, 1905. 8vo. 654 pp. Cloth. $4.50

PIERPONT (J.). On multiple integrals. 4to. 1905. (*Transactions of the American Mathematical Society*, 6, pp. 416–434.)

RÉTHY (M.). See BOLYAI.

SCHMIDT (E.). Entwicklung willkürlicher Funktionen nach Systemen vorgeschriebener. (Diss.) Göttingen, 1905.

SCHRÖDER (R.). Die Anfangsgründe der Differentialrechnung und Integralrechnung. Für Schüler von höheren Lehranstalten und Fachschulen und zum Selbstunterricht. Mit zahlreichen Uebungsbeispielen. Leipzig, Teubner, 1905. 8vo. 7 + 131 pp. M. 1.40

STOLZ (O.) und GMEINER (J. A.). Einleitung in die Funktionentheorie. 2te, umgearbeitete und vermehrte Auflage der von den Verfassern in der "Theoretischen Arithmetik" nicht berücksichtigten Abschnitte der "Vorlesungen über allgemeine Arithmetik" von O. Stolz. (In 2 Abteilungen.) Abteilung II. Leipzig, Teubner, 1905. 8vo. 8 + 356 pp.

TÖTÖSSY DE ZEPETHNEK (B.). See BOLYAI.

WIENER (H.). Sammlung mathematischer Modelle, herausgegeben von der Verlagsbuchhandlung B. G. Teubner. Leipzig, Teubner, 1905. 8vo. 28 pp.

WORKED EXAMPLES in coordinate geometry. Graduated course in right line and circle. 2d edition. London, Clive, 1905. 12mo. 80 pp. (University tutorial series.) 1s. 6d.

II. ELEMENTARY MATHEMATICS.

ADAMS (H.). Practical trigonometry, for engineers, architects, surveyors. Enlarged edition. London, Whittaker, 1905. 12mo. 78 pp. 2s. 6d.

BÖTTGER (A.) und HARTENSTEIN (H.). Die Trigonometrie. Für den Unterricht an Realschulen bearbeitet. Leipzig, Dürr, 1906. 8vo. 35 pp.

BOS (H.). Géométrie élémentaire. 20e édition. Paris, Hachette, 1905. 16mo. 288 pp. Fr. 2.00

Burali-Forti (C.). Aritmetica e geometria per la quarta classe elementare. Torino, Paravia, 1905. 16mo. 100 pp. L. 0.60

Catania (S.). Aritmetica razionale ad uso delle scuole secondarie superiori. Catania, Giannotta, 1904. 16mo. 184 pp. L. 1.60

Euclid. Elements, book I. With commentary based principally upon that of Proclus Diadochus, by W. B. Frankland. Cambridge, University Press, 1905. 8vo. 156 pp. Cloth. 6s.

Fine (H. B.). A college algebra. Boston, Ginn, [1905]. 8vo. 8 + 595 pp. 12mo. Cloth. $1.50

Font (N.) et Hugon (C.). Algèbre. Compléments à l'usage des élèves du deuxième cycle (sections C et D) (programmes du 31 mai 1902). Marseille, Ferran, 1905. 16 mo. 8 + 150 pp.

Frankland (W. B.). See Euclid.

Gazzaniga (P.). See Veronese (G.).

Girod (J.). Précis de trigonométrie, conforme aux programmes du 31 mai 1902, pour les classes de seconde C et D, de première C et D, et de mathématiques A et B. Avec 414 problèmes et exercices proposés. Paris, Alcan, 1905. 8vo. 8 + 210 pp. Fr. 2.40

Hall (H. S.) and Stevens (F. H.). A school geometry. Part I: Lines and angles; rectilineal figures. London, Macmillan, 1905. 12mo. Cloth. 1s.

Hall (W.). Tables and constants to four figures. For use in technical, physical, and nautical computation, and adapted to the requirements of junior mathematical students. Cambridge, University Press (New York, Macmillan), 1905. 8vo. 9 + 60 pp. Cloth. $0.90

Hartenstein (H.). See Böttger (A.).

Heegaard (P.). Lærebog i matematik. Kjöbenhavn, 1905. 8vo. Kr. 7.00

Hugon (C.). See Font (N.).

Jacob (J.). Lehrbuch der Arithmetik für Untergymnasien. Abteilung II: Lehrstoff der 3. und 4. Klasse. Wien, Deuticke, 1905. 8vo. 3 + 142 pp. Cloth. M. 1.80

Jacobsthal (W.). See Weber (H.).

Lightfood (J.). Elementary algebra for junior students, with exercises and answers. Part I of an Elementary and intermediate algebra. London, Ralph and Holland, 1905. 12mo. 228 pp. Cloth. 2s. 6d.

Manuel d'algèbre et de trigonométrie. Par une réunion de professeurs. Paris, Poussielgue, [1905]. 16mo. 8 + 216 pp. (Collection d'ouvrages classiques rédigés en cours gradués.)

Mayer (J. E.). Das mathematische Pensum des Primaners. Ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien, für Techniker etc., sowie besonders für das Selbststudium. Heft 17–18: Lehre von den Kegelschnitten in elementarer Behandlung; vollständig gelöste Uebungsbeispiele; Elemente der projektiven Geometrie. Leipzig, Schäfer, 1905. 8vo. 83 pp. M. 2.00

Model answers to intermediate pure mathematics papers set at the London University intermediate examination in arts and science from 1897 to 1905. London, Clive. 1905. 12mo. 174 pp. (University tutorial series.) 2s. 6d.

SISK (B. F.). The foundations of higher arithmetic. New York, Silver, Burdette & Co., [1905]. 12mo. 10 + 203 pp. Cloth. (Standard series.) $0.65

SMITH (D. E.). Algebra for beginners. Boston, Ginn, [1905]. 12mo. 6 + 154 + 27 pp. Cloth. $0.50

SOMMERVILLE (F. H.). First year in algebra. New York, American Book Company [1905]. 12mo. 208 pp. Cloth. $0.60

STEVENS (F. H.). See HALL (H. S.).

VERONESE (G.). Elementi di geometria ad uso dei ginnasi e licei e istituti tecnici (primo biennio), trattati con la collaborazione di P. Gazzaniga. Parte I. 3a edizione. Verona e Padova, Drucker, 1904. 8vo. 24 + 124 pp. L. 1.25

WEBER (H.) und WELLSTEIN (J.). Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. (In 3 volumes.) Vol. II: Elementare Geometrie. Bearbeitet von H. Weber, J. Wellstein und W. Jaçobsthal. Leipzig, Teubner, 1905. 8vo. 12 + 604 pp.

WELLSTEIN (J.). See WEBER (H.).

WILLIS (H. G.). Elementary modern geometry, experimental and theoretical. Part I (Chapters I–IV): Triangles and parallels. London, Frowde, 1905. 12mo. 244 pp. Cloth. 2s.

ZOLESE (C.). Trigonometria piana e sferica. Libro di testo per le scuole secondarie superiori. Napoli, De Rubertis, 1905. 16mo. 78 pp.

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. Vol. II: Elekromagnetische Theorie der Strahlung. Leipzig, Teubner, 1905. 8vo. 10 + 406 pp. Cloth. M. 10.00

BARKER (A. H.). Graphic methods of engine design, including the graphical treatment of the balancing of engines. 2d edition. London, Technical Pub. Co., 1905. 12mo. 218 pp. Cloth. 3s. 6d.

BAUERSFELD (W.). Die automatische Regelung der Turbinen. Berlin, Springer, 1905. 8vo. 7 + 208 pp. M. 6.00

BERG (E.). See CHWOLSON (O. D.).

BRUNN (A. VON). Die Säkularbeschleunigung des Mondes. (Diss.) Göttingen, 1905. 8vo. 100 pp.

CHWOLSON (O. D.). Lehrbuch der Physik. Vol. III: Die Lehre von der Wärme. Uebersetzt von E. Berg. Braunschweig, Vieweg, 1905. 8vo. 11 + 988 pp. M. 16.00

FÖLZER und KRAUS. Hydromechanik oder die Lehre vom Gleichgewicht und der Bewegung des Wassers. Strelitz, 1905. 8vo. 62 pp. M. 4.00

GORTON (F. R.). Ueber das Minimum bei Spitzenentladung in Chlor, Brom und Jod und die Empfindlichkeit von Spitzen. (Diss.) Berlin, 1905. 8vo. 37 pp.

HATT (W. K.). Flexure of reinforced concrete beams. Chicago, Western Society of Engineers, 1904. 8vo. 14 pp. $0.50

HIROI (I.) The statically-indeterminate stresses in frames commonly used for bridges. New York, Van Nostrand, 1905. 12mo. 9 + 174 pp. Cloth. $2.00

HOTOPP (L.). See KECK (W.).

HOWE (M. A.) Cross-bending tests on steel-concrete beams. Chicago, Western Society of Engineers, 1904. 8vo. 15 pp. $0.50

KECK (W.). Vorträge über Elastizitäts-Lehre als Grundlage für die Festigkeitsberechnung der Bauwerke. 2te, vermehrte Auflage, neu bearbeitet von S. Hotopp. Teil I. Hannover, Helwing, 1905. 8vo. 8 + 306 pp. M. 8.00

KRAUS. See FÖLZER.

LIEBENOW (C.). Ueber die Abhängigkeit der Kapazität der Bleiakkumulatoren von der Stromstärke. (Diss.) Göttingen, 1905. 8vo. 32 pp.

PEDROTTI (L.). Elementi di geometria descrittiva. Milano, Sonzogno, 1905. 16mo. 61 pp. (Biblioteca del popolo, No. 374.) L. 0.15

PORTER (A. W.). An intermediate course of mechanics. London, Murray, 1905. 12mo. 430 pp. Cloth. 5s.

PULLEN (W. W. F.). Application of graphic and other methods to the design of structures. 2d edition. London, Technical Pub. Co., 1905. 12mo. 35 pp. Cloth. 5s.

RIEBESELL (P.). Ueber den Kurzschluss der Spulen und die Vorgänge bei der Kommutation des Stromes eines Gleichstromankers. Ein Beitrag zur Theorie der Dynamomaschine. (Diss.) Kiel, 1905. 8vo. 62 pp.

ROSENBERG (F.). First stage mechanics of solids. 5th edition. London, Clive, 1905. 12mo. 288 pp. Cloth. (Organic science series.) 2s.

SILBERSTEIN (L.). Wyklady zakopańskie z dziedziny fizyki teoretycznéj. Warszawa, 1905. 8vo. 2 + 104 pp. R. 1.20

UNDEUTSCH (H.). Theorie, Konstruktion, Prüfung und Regelung der Fallbremsen und Energie-Indikatoren, einschliesslich der Beanspruchung und Prüfung der Schachtförderseile auf Stoss, Freifall-, Fang- und Indikator-Versuche. Wien, Deuticke, 1905. 8vo. 11 + 244 pp. M. 10.00

WINKELMANN (A.). Handbuch der Physik. Vol. IV, 2te Hälfte: Elektrizität und Magnetismus. Leipzig, Barth, 1905. 8vo. Pp. 385–1014. M. 20.00

LIBRARY

ON A FAMILIAR THEOREM OF THE THEORY OF FUNCTIONS.

BY PROFESSOR EDMUND LANDAU.

IN the second part of a paper* published in 1896, Professor Osgood considers the theorem:

"Let a function $f(z)$ of the complex variable z be single-valued and analytic at all points of the neighborhood of the point $z = c$ (exclusive of $z = c$), that is, for $0 < |z - c| < \gamma$; also, for these values of z, let $|f(z)|$ remain less than some fixed value. Then $\lim_{z=c} f(z) = \alpha$ exists, and if at $z = c$ we assign to the function the value α, then $f(z)$ will be analytic at the point c."

This follows easily, for example, from Laurent's theorem. Professor Osgood mentions the various proofs to be found in the literature of the subject and states that in several treatises the following fallacious proof is contained:

Let a function $\phi(z)$ be defined by the equations

$$\phi(z) = (z - c)f(z)(0 < |z - c| < \gamma), \quad \phi(c) = 0;$$

then, by hypothesis, $\lim_{z=c} \phi(z) = 0$, and $\phi(z)$ is continuous at the point c and analytic for $0 < |z - c| < \gamma$. From this some writers incorrectly infer that $\phi(z)$ is also analytic at $z = c$. This conclusion is not justified, even when the existence of $\lim_{z=c} f(z) = \alpha$, or of

$$(1) \qquad \phi'(c) = \lim_{z=c} \frac{(z - c)f(z)}{z - c} = \alpha$$

is presupposed.† For from the existence of the derivative of $\phi(z)$, without knowledge of its continuity, it could not be inferred, previously to 1900, that $\phi(z)$ is analytic. Not until Goursat's ‡ proof of Cauchy's integral theorem had appeared was this conclusion, to which Professor Osgood quite rightly took exception in 1896, later justified.

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* "Some points in the elements of the theory of functions." BULLETIN, ser. 2, vol. 2, pp. 296–302.

† This assumption is no restriction of the generality. For the function $(z - c)f(z)$ has a limiting value, namely zero, for $z = c$, and it is sufficient to show that this function is analytic at $z = c$.

‡ "Sur la définition générale des fonctions analytiques d'après Cauchy." *Trans. Amer Math. Society*, vol 1 (1900), pp. 14–16. Compare the more correct presentation of Goursat's proof by Pringsheim, "Ueber den Goursatschen Beweis des Cauchyschen Integralsatzes," ibid, vol. 2 (1901), pp. 413–421.

At the end of his paper, Professor Osgood has added to the valid proofs of the theorem two new demonstrations.

I should like to communicate another proof, which stands in closer relation to the fallacious process of thought previously referred to, and which permits the gap to be filled without the use of Goursat's theorem.

There was lacking only the proof that the derivative $\phi'(z)$ is continuous at the point $z = c$. Since for $0 < |z - c| < \gamma$ we have

$$\phi'(z) = (z - c)f'(z) + f(z), \tag{2}$$

and since the existence of

$$\lim_{z=c} f(z) = \alpha \tag{3}$$

might be assumed, it only remains to show that

$$\lim_{z=c} (z - c)f'(z)$$

exists and is equal to zero.

From (3), there exists for any previously chosen δ an ϵ such that for $0 < |z - c| < \epsilon$ we have $|f(z) - \alpha| < \delta$. If we take $0 < |z - c| < \frac{2}{3}\epsilon$, and apply Cauchy's formula to the function $f(z) - \alpha$, we get

$$f'(z) = \frac{1}{2\pi i}\int \frac{f(x) - \alpha}{(x - z)^2}\,dx,$$

where the integral is extended over the circumference with center at z and radius $|z - c|/2$. Hence follows

$$|f'(z)| < \frac{1}{2\pi}\cdot 2\pi\frac{|z - c|}{2}\cdot\frac{\delta}{\frac{|z - c|^2}{4}} = \frac{2\delta}{|z - c|},$$

$$|(z - c)f'(z)| < 2\delta,$$

or

$$\lim_{z=c} (z - c)f'(z) = 0$$

and, by use of (1), (2) and (3),

$$\lim_{z=c} \phi'(z) = \alpha = \phi'(c).$$

BERLIN,
August 9, 1905.

RATIONAL PLANE CURVES RELATED TO RIEMANN TRANSFORMATIONS.

BY PROFESSOR H. S. WHITE.

(Read before the American Mathematical Society, September 8, 1905.)

In a plane considered as filled with algebraic curves of all orders and deficiencies, a perfectly arbitrary net or doubly infinite linear system of such curves may be regarded as effecting a one-to-one transformation. For exceptional curves the transformation may be two-to-one (or m-to-one), and such curves are termed in another connection involution curves of the net. Only in case the net consists of rational curves, having but a single variable point of intersection is the transformation a Cremona transformation. In other cases, barring all involution curves of the net, it is called a Riemann transformation, or a birational transformation of the curve under consideration. Let the x-plane be transformed into the y-plane by the substitution

$$\rho y_1 = \phi_1(x_1, x_2, x_3) = \phi_1(x), \quad \rho y_2 = \phi_2(x), \quad \rho y_3 = \phi_3(x);$$

and let $f(x) = 0$ represent a curve which is transformed into $F(y) = 0$. To every point (y) on the curve F (with a finite number of exceptions) let only one point (x) correspond, then the transformation is birational for that curve; and we may call it a Riemann transformation of the plane, as proposed by Dr. Kasner. If the curves $\phi_1(x) = 0$, $\phi_2(x) = 0$, $\phi_3(x) = 0$ are rational of order n, with multiple points of like structure at fixed points, and enough other fixed simple base points to leave only one intersection variable, then we have a Cremona transformation of the entire plane. My purpose is to call attention to a peculiarity of rational curves under Riemann transformations, viz., to show that each possesses, for every Riemann transformation, one of the Cremona type which is on that curve equivalent to the former.

Begin with a straight line U, or $(ux) = 0$. Replace the Riemann substitution by the following:

$$\begin{aligned} \rho y_1 &= \phi_1(x) + \psi_1(x)\cdot(ux) \equiv \Phi_1(x), \\ \rho y_2 &= \phi_2(x) + \psi_2(x)\cdot(ux) \equiv \Phi_2(x), \\ \rho y_3 &= \phi_3(x) + \psi_3(x)\cdot(ux) \equiv \Phi_3(x), \end{aligned}$$

and take ψ_1, ψ_2, ψ_3 any forms of order $n-1$ to make the expressions on the right homogeneous. For every point x on $(ux)=0$ this gives the same point (y) as before, *i. e.*, it is equivalent to the former on the line U. The net of Φ-curves can now be made a Cremona net of Jonquières type. For each ψ contains $\frac{1}{2}n(n+1)$ constants linearly. To have a point of multiplicity $n-1$ in a specified point O is equivalent to $\frac{1}{2}(n-1)n$ linear conditions. Each ψ retains now n parameters: select those arbitrarily for one ψ, say ψ_1, then dispose of the remaining $2n$ so that the curves $\Phi_2=0$ and $\Phi_3=0$ may have on $\Phi_1=0$ $2n-2$ additional points of intersection in common, none of them on the line U. The transformation is now of the Cremona type, as was required.

On every straight line, a Riemann transformation is conformable with an indefinitely great number of Cremona transformations.

Next consider any rational curve of order above the first, $f(x)=0$. Find a Cremona transformation S which converts f into the line U, and let S^{-1} denote its inverse. Let R denote the proposed Riemann transformation, call $SR \equiv R'$, and by the above theorem R' must be conformable along the line U to a Cremona transformation K. Then along the curve f, we must have $S^{-1}SR$ or $S^{-1}R'$ conformable with $S^{-1}K$ or K'; and evidently this is of the Cremona type, since Cremona transformations constitute a group. This shows that:

On any rational plane curve, not an involution curve of the net, a Riemann transformation R with a net of φ-curves is equivalent to or conformable with a Cremona transformation whose net of Φ-curves is of the Jonquières type.

Remembering that a rational curve has no invariants (or moduli) under algebraic transformation, we have learned that also it can have none under Cremona transformations; or negatively stated, the distinctive properties of the Cremona group within the group of Riemann transformations must be sought in their effect on systems of curves or on single curves of deficiency higher than zero.

The extension of this remark to space of more than two dimensions is not apparent. For verification, nets of conics and cubics have been employed, and no obstacle found to the actualization of the steps referred to in the above counting of constants.

EVANSTON, ILLINOIS,
June, 1905.

ON LAMÉ'S SIX EQUATIONS CONNECTED WITH TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

BY J. E. WRIGHT, M.A.

LAMÉ* has shown for a triply orthogonal system of surfaces given by the parameters ρ, ρ_1, ρ_2 that if the square of the element of length is given by $ds^2 = H^2 d\rho + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2$, where H, H_1, H_2 are certain functions of ρ, ρ_1, ρ_2, then H, H_1, H_2 must satisfy the following system of equations:

$$\frac{\partial^2 H}{\partial \rho_1 \partial \rho_2} = \frac{1}{H_1}\frac{\partial H}{\partial \rho_1}\frac{\partial H_1}{\partial \rho_2} + \frac{1}{H_2}\frac{\partial H}{\partial \rho_2}\frac{\partial H_2}{\partial \rho_1} \tag{1}$$

and two others of the same type; (2), (3)

$$\frac{\partial}{\partial \rho_1}\left(\frac{1}{H_1}\frac{\partial H}{\partial \rho_1}\right) + \frac{\partial}{\partial \rho}\left(\frac{1}{H}\frac{\partial H_1}{\partial \rho}\right) + \frac{1}{H_2^2}\frac{\partial H}{\partial \rho_2}\frac{\partial H_1}{\partial \rho_2} = 0 \tag{4}$$

with two others of this type. (5), (6)

Also if V is a function of x, y, z for which

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

he has shown that

$$\frac{\partial}{\partial \rho}\left(\frac{H_1 H_2}{H}\frac{\partial V}{\partial \rho}\right) + \frac{\partial}{\partial \rho_1}\left(\frac{H_2 H}{H_1}\frac{\partial V}{\partial \rho_1}\right) + \frac{\partial}{\partial \rho_2}\left(\frac{H H_1}{H_2}\frac{\partial V}{\partial \rho_2}\right) = 0.$$

If the system of coordinates ρ, ρ_1, ρ_2 is isothermal, this equation must be satisfied by $V = \rho$, or by $V = \rho_1$, or by $V = \rho_2$. Hence $H_1 H_2 / H = Q^2$, where Q is a function of ρ_1 and ρ_2 only. Similarly $H_2 H / H_1 = Q_1^2$, and $H H_1 / H_2 = Q_2^2$, where Q_i is a function not involving the variable ρ_i. Hence $H = Q_1 Q_2$, $H_1 = Q_2 Q$, $H_2 = Q Q_1$. The six equations given above transform into six in the variables Q. Lamé † gives a solution of these equations by first finding a solution of equations (1), (2), (3) and using this to solve the remainder. He makes the statement that his solution of the first three equations is the most general possible * ; this statement is obviously inaccurate and it seems of interest to give a complete solution of the equations. The solution of the first three equations, or rather a comparison of two different solutions, leads to a curious result in the theory of elimination.

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* Leçons sur les coordonées curvilignes (1859), pp. 76, 78.

† Loc. cit., p. 99.

Equation (1) becomes

$$Q\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} = Q_1\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2} + Q_2\frac{\partial Q_1}{\partial \rho_2}\frac{\partial Q}{\partial \rho_1}, \tag{7}$$

with similar expressions for (2) and (3). Multiply (7) by $\partial Q_1/\partial\rho$, and the transformed expression for (3) by $\partial Q_1/\partial\rho_2$, and add. The result is

$$Q_1\left[\frac{\partial Q_2}{\partial \rho}\frac{\partial Q}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} + \frac{\partial Q_1}{\partial \rho}\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2}\right] = 0.$$

Hence unless all the Q's vanish identically

$$\frac{\partial Q_2}{\partial \rho}\frac{\partial Q}{\partial \rho_1}\frac{\partial Q_1}{\partial \rho_2} + \frac{\partial Q_1}{\partial \rho}\frac{\partial Q_2}{\partial \rho_1}\frac{\partial Q}{\partial \rho_2} = 0. \tag{8}$$

Assume that none of the derivatives in (8) vanish identically and write

$$K = -\frac{\partial Q}{\partial \rho_1}\Big/\frac{\partial Q}{\partial \rho_2}, \quad K_1 = -\frac{\partial Q_1}{\partial \rho_2}\Big/\frac{\partial Q_1}{\partial \rho}, \quad K_2 = -\frac{\partial Q_2}{\partial \rho}\Big/\frac{\partial Q_2}{\partial \rho_1}.$$

Equation (8) becomes $KK_1K_2 = 1$, where K_i is a function not involving ρ_i. By taking logarithms and differentiating with respect to two of the variables ρ, it is easy to prove that the most general values for the K's are

$$K = \frac{a_2}{a_1}, \quad K_1 = \frac{a}{a_2}, \quad K_2 = \frac{a_1}{a},$$

where a is a function of ρ only and similarly a_1 and a_2 are functions of ρ_1 and ρ_2 alone respectively.

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* Loc. cit., p. 100, line 23.

If we put $\int a d\rho = \sigma$, with similar values for σ_1 and σ_2, we deduce that Q is a function of $\sigma - \sigma_2$ only, Q_1 a function of $\sigma_2 - \sigma$ only, and Q_2 a function of $\sigma - \sigma_1$ only. Substitute these values in (7) and it readily reduces to

$$\frac{Q'}{Q'} + \frac{Q_1}{Q_1'} + \frac{Q_2}{Q_2'} = 0.$$

Hence

$$\frac{Q}{Q'} = n(\sigma_1 - \sigma_2), \quad \frac{Q_1}{Q_1'} = n(\sigma_2 - \sigma), \quad \frac{Q_2}{Q_2'} = n(\sigma - \sigma_1),$$

where n is a constant, and therefore

$$Q = c(\sigma_1 - \sigma_2)^{1/n}, \quad Q_1 = c_1(\sigma_2 - \sigma)^{1/n}, \quad Q_2 = c_2(\sigma - \sigma_1)^{1/n}, \qquad (A)$$

where c, c_1, c_2 are constants. This is the most general solution, provided none of the derivatives in (8) vanish. If, however, for example, $\partial Q_2/\partial\rho_1 = 0$, then either $\partial Q_2/\partial\rho = 0$, or $\partial Q/\partial\rho_1 = 0$, or $\partial Q_1/\partial\rho_2 = 0$. Equation (7), however, shows that $\partial Q_2/\partial\rho_1 = 0$ implies either $\partial Q/\partial\rho_1 = 0$ or $\partial Q_1/\partial\rho_2 = 0$. $\partial Q_2/\partial\rho_1 = 0$, $\partial Q_1/\partial\rho_2 = 0$ lead to the solution

$$Q_1 = f(\rho), \quad Q_2 = cf(\rho), \quad Q = \phi(\rho_1, \rho_2), \qquad (B)$$

where f and ϕ are arbitrary functions of their arguments and c is a constant. $\partial Q_2/\partial\rho_1 = 0$, $\partial Q/\partial\rho_1 = 0$ lead to the solution

$$Q_2 = f(\rho), \quad Q = \phi(\rho_2), \quad Q_1 = F(Q, Q_2), \qquad (C)$$

where F is homogeneous and of unit degree in Q_1, Q_2. These types (A), (B), (C) are the only three types of solution.

We now proceed to the integration of the equations in a different manner. Write $\log Q = \lambda$, $\log Q_1 = \lambda_1$, $\log Q_2 = \lambda_2$. Equations (1), (2), (3) become

$$\frac{\partial\lambda_2}{\partial\rho_1}\frac{\partial\lambda_1}{\partial\rho_2} = \frac{\partial\lambda_2}{\partial\rho_1}\frac{\partial\lambda}{\partial\rho_2} + \frac{\partial\lambda}{\partial\rho_1}\frac{\partial\lambda_1}{\partial\rho_2} \qquad (9)$$

and two similar equations.

Write $\lambda_1 - \lambda_2 = \omega$, $\lambda_2 - \lambda = \omega_1$, $\lambda - \lambda_1 = \omega_2$ and (9) becomes

$$\frac{\partial\omega_1}{\partial\rho_1}\frac{\partial\omega_2}{\partial\rho_2}-\frac{\partial\omega_1}{\partial\rho_2}\frac{\partial\omega_2}{\partial\rho_1}=0,$$

or

$$J\begin{pmatrix}\omega_1 & \omega_2\\ \rho_1 & \rho_2\end{pmatrix}=0.$$

Hence the equation implies the existence of a relation, $f(\omega_1, \omega_2, \rho)=0$. Exactly similarly there are relations

$$f_1(\omega_2, \omega, \rho_1)=0, \quad f_2(\omega, \omega_1, \rho_2)=0.$$

From these equations, together with $\omega+\omega_1+\omega_2=0$, it is easy to deduce that either ρ, ρ_1, ρ_2 do not any of them occur explicitly in f, f_1, f_2 or if, for example, ρ occurs in f, it is easy to prove that either ρ_1 is absent from f_1, or ρ_2 from f_2. The latter case is thus reduced to the former, for if one relation of the type $f(\omega, \omega_1)=0$ exist, then two others of that type also exist in virtue of the relation $\omega+\omega_1+\omega_2=0$.

Substituting for the ω's in terms of the Q's, we immediately deduce that the solution is equivalent to the statement that a homogeneous relation $F(Q, Q_1, Q_2)=0$ exists among the Q's.

Combining the two solutions we have the following theorem connected with the theory of elimination:

Let $F(Q, Q_1, Q_2)=0$ *be any homogeneous relation. It is possible to express* Q *as a function of two variables* ρ_1 *and* ρ_2, Q_1 *as a function of two variables* ρ_2 *and* ρ, *and* Q_2 *as a function of the variables* ρ *and* ρ_1, *in two cases only:*

(*A*) *If* F *is of the form* $aQ^n+a_1Q_1^n+a_2Q_2^n$, *where* a, a_1, a_2 *are constants.*[*]

(*C*) *If* F *is general, and e. g.* Q *is a function of* ρ_1 *only, and* Q_1 *a function of* ρ *only.*

It is not difficult to complete the solution of the equations (1), · · ·, (6). It may readily be shown that for case (*A*) (4), (5), and (6) are not satisfied unless $n=\frac{1}{2}$, and then

$$\sigma=A\wp\left(\frac{\rho+a}{\sqrt{c}}, g_2, g_3\right)+B,$$

$$\sigma_1=A\wp\left(\frac{\rho_1+a_1}{\sqrt{c_1}}, g_2, g_3\right)+B,$$

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[*] This includes (*B*).

$$\sigma_2 = A\wp\left(\frac{\rho_2 + a_2}{\sqrt{c_2}}, g_2, g_3\right) + B,$$

where the a's, the g's, A and B are constants.

For case (c) the complete solution is

$$Q_1 = a \operatorname{cosec} (b\rho_2 + c), \quad Q_2 = a' \operatorname{cosec} (b'\rho_1 + c'),$$
$$Q = A[\operatorname{cosec}^2 (b\rho_2 + c) - \operatorname{cosec}^2 (b'\rho_1 + c')]^{\frac{1}{2}},$$

where A, a, a', b, b', c, c' are constants such that

$$a^2b'^2 + a'^2b^2 = 0.$$

In case (B)

$$Q_1 = \frac{1}{a\rho + b}, \quad Q_2 = \frac{k}{a\rho + b},$$

where a, b, k are constants, and Q is a function of ρ_1 and ρ_2 which satisfies the equation

$$\left(k^2\frac{\partial^2}{\partial\rho_2^2} + \frac{\partial^2}{\partial\rho_1^2}\right)\log Q + a^2Q^2 = 0.$$

Of these three types of solution, the first is the same as that given by Lamé.* He gives it in different form, and his method of obtaining it is different. He falls into the error of imagining that the most general solution of the *first* three of his equations corresponds to the case of $n = \frac{1}{2}$, and it happens that this error is largely corrected because the *second* three equations require this limitation in case (A); case (B), however, escapes his notice.

The surfaces corresponding to the three solutions are readily obtained. (A) gives a system of confocal quadrics, and (C) a system of confocal spheroids with their axial planes. (B) gives a system of concentric spheres, and the conical surfaces obtained by joining the common centre to any set of isothermal lines on one of them.

BRYN MAWR,
October, 1905.

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* Loc. cit., p. 104.

CERTAIN SURFACES ADMITTING OF CONTINUOUS DEFORMATION WITH PRESERVATION OF CONJUGATE LINES.

BY DR. BURKE SMITH.

(Read before the American Mathematical Society, December 30, 1904.)

BIANCHI has shown* that surfaces which admit of continuous deformation with preservation of conjugate lines are the associates of surfaces whose curvature in terms of the parameters of the asymptotic lines is of the form $K = -[\phi(u) + \chi(v)]^{-2}$, where ϕ and χ are arbitrary. Stäckel † has considered those surfaces on which the two families of lines of curvature fall together into a single family which are also at the same time minimal lines and asymptotic lines. The curvature of such surfaces may be written in the form $K = e^{-2m(v)}$ and consequently it is of the above type. We proceed to find the coordinates x_0, y_0, z_0 of certain of these surfaces.

The fundamental magnitudes of the Gauss sphere which corresponds to such a surface may be written

$$(1)\quad e = 0, \quad f = \frac{-2\phi'(u)\psi'(v)}{[\phi(u) + \psi(v)]^2}, \quad g = \frac{-2m'(v)\psi'(v)}{\phi(u) + \psi(v)} + n(v),$$

where the primes denote derivatives and ϕ, ψ, m and n are arbitrary. We consider only the case where $\phi \equiv u$ and $\psi \equiv v$. Upon substituting in the fundamental equations for the second derivatives of X and the first derivatives of x with regard to u and v we have

$$(2)\qquad \frac{\partial^2 X}{\partial u^2} = \frac{-2}{u+v}\frac{\partial X}{\partial u},$$

$$(3)\qquad \frac{\partial^2 X}{\partial u \partial v} = \frac{-m'}{2}\frac{\partial X}{\partial u} + \frac{2}{(u+v)^2} X,$$

$$(4)\quad \frac{\partial^2 X}{\partial v^2} = \begin{Bmatrix} 2\ 2 \\ 1 \end{Bmatrix}' \frac{\partial X}{\partial u} + \tfrac{1}{2}\left[m' - \frac{4}{u+v}\right]\frac{\partial X}{\partial v} + \left[\frac{2m'}{u+v} - n\right] X,$$

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* Bianchi-Lukat: Vorlesungen über Differential-Geometrie, p. 337.

† *Leipziger Ber.*, vol. 48, p. 479.

$$(5)\quad \begin{cases} \dfrac{\partial x_0}{\partial u} = -e^m \dfrac{\partial X}{\partial u}, \\[2ex] \dfrac{\partial x_0}{\partial v} = \dfrac{u+v}{2}\left[2m' - (u+v)n\right] e^m \dfrac{\partial X}{\partial u} - e^m \dfrac{\partial X}{\partial v}, \end{cases}$$

with similar equations for Y, Z, y_0, z_0, where

$$\begin{Bmatrix} 2 & 2 \\ & 1 \end{Bmatrix}' = \tfrac{1}{4}\big(2(u+v)(m'' - m'^2) + m'[6 + (u+v)^2 n] \\ - (u+v)[(u+v)n' + 4n]\big).$$

The general integral of (2) is

$$(6)\qquad X = \frac{F(v)}{u+v} + G(v).$$

Now (3) has equal invariants, and by the substitution $\theta = X\sqrt{ie^m}$ becomes $\dfrac{\partial^2\theta}{\partial u \partial v} = \dfrac{2\theta}{(u+v)^2}$, of which the general integral is

$$\theta = \frac{2(U_1 + V_1)}{u+v} - U_1' - V_1',$$

U_1 and V_1 being arbitrary functions of u and v respectively. Comparing with (6), we must have $U_1 \equiv 0$ and X can be written in the form

$$X = \frac{2V}{u+v} - V' - \frac{m'}{2} V.$$

Since also (4) must be satisfied, we have to determine V the equation

$$(7)\quad V''' + \left(m'' - \frac{m'^2}{4} + n\right) V' + \left(\frac{m'''}{2} - \frac{m''m'}{4} + \frac{n'}{2}\right) V = 0,$$

where the primes denote derivatives. To every solution of (7) we have a value for X, and similar equations hold for Y and Z. When $n = m'^2/4 - m'' + 1$, (7) takes the form $V''' + V' = 0$ and consequently, $V = a \cos v + b \sin v + c$. If the constants a, b, c, are so chosen that $X^2 + Y^2 + Z^2 = 1$ then (5) will give by quadrature the coördinates x_0, y_0, z_0. Take, for example, a, b, c, so that

$$X = \left(\frac{2}{u+v} - \frac{m'}{2}\right)\cos v + \sin v,$$

$$Y = \left(\frac{2}{u+v} - \frac{m'}{2}\right)\sin v - \cos v, \qquad Z = i\left(\frac{2}{u+v} - \frac{m'}{2}\right).$$

Then from (5) we have

$$(8)\begin{cases} x_0 = \dfrac{-2e^m \cos v}{u+v} + \displaystyle\int e^m \left[\left(\frac{m'^2}{4} - \frac{m''}{2}\right)\cos v - \frac{m'}{2}\sin v\right]dv \\ y_0 = \dfrac{-2e^m \sin v}{u+v} + \displaystyle\int e^m \left[\left(\frac{m'^2}{4} - \frac{m''}{2}\right)\sin v + \frac{m'}{2}\cos v\right]dv \\ iz_0 = \dfrac{2e^m}{u+v} - \displaystyle\int e^m \left(\frac{m'^2}{4} - \frac{m''}{2} + 1\right)dv, \end{cases}$$

where $m = m(v)$ may be chosen arbitrarily.

In another place the writer has shown that the equations of the surfaces $S(x, y, z)$ which are associate to a given surface $S_0(x_0, y_0, z_0)$ may be written in the form

$$(9)\quad \begin{cases} x = \rho^{-\frac{1}{2}}(eg - f^2)^{-\frac{1}{2}}\left(\theta \dfrac{\partial^2 x_0}{\partial u \partial v} - \dfrac{\partial\theta}{\partial u}\dfrac{\partial x_0}{\partial v} - \dfrac{\partial\theta}{\partial v}\dfrac{\partial x_0}{\partial u}\right), \\ y = \rho^{-\frac{1}{2}}(eg - f^2)^{-\frac{1}{2}}\left(\theta \dfrac{\partial^2 y_0}{\partial u \partial v} - \dfrac{\partial\theta}{\partial u}\dfrac{\partial y_0}{\partial v} - \dfrac{\partial\theta}{\partial v}\dfrac{\partial y_0}{\partial u}\right), \\ z = \rho^{-\frac{1}{2}}(eg - f^2)^{-\frac{1}{2}}\left(\theta \dfrac{\partial^2 z_0}{\partial u \partial v} - \dfrac{\partial\theta}{\partial u}\dfrac{\partial z_0}{\partial v} - \dfrac{\partial\theta}{\partial v}\dfrac{\partial z_0}{\partial u}\right), \end{cases}$$

where $-1/\rho^2$ is the curvature of S_0 and θ is a solution of the equation

$$\frac{\partial^2\theta}{\partial u \partial v} = \left(-\frac{1}{4}\frac{\partial\rho}{\partial u}\frac{\partial\rho}{\partial v} - f\right)\theta.$$

This equation becomes, in the case of the surfaces defined by (8), $\dfrac{\partial^2\theta}{\partial u \partial v} = \dfrac{2\theta}{(u+v)^2}$, of which the general solution is

$$\theta = \frac{2(U+V)}{u+v} - U' - V'.$$

On substituting in (9), we have

$$x = e^{-m/2}(u+v)^2 \left\{ \left[A\left(\frac{m'^2}{4} - \frac{m''}{2}\right) + Bm' + D \right] \cos v \right.$$
$$\left. - \left(A\frac{m'}{2} + B\right) \sin v \right\},$$

$$y = e^{-m/2}(u+v)^2 \left\{ \left[A\left(\frac{m'^2}{4} - \frac{m''}{2}\right) + Cm' + D \right] \sin v \right.$$
$$\left. + \left(A\frac{m'}{2} + C\right) \cos v \right\},$$

$$-iz = e^{-m/2}(u+v)^2 \left\{ A\left(\frac{m'^2}{4} - \frac{m''}{2} + 1\right) + Cm' + D \right\},$$

where

$$A = \frac{U+V}{(u+v)^2} - \frac{U'}{u+v} + \frac{U''}{2}, \quad B = \frac{U'-V'}{(u+v)^2} + \frac{U''}{u+v},$$

$$C = \frac{U'-V'}{(u+v)^2} - \frac{U''}{u+v}, \quad D = \frac{U''+V''}{(u+v)^2}.$$

These equations, depending on three arbitrary functions, U, V and m, are therefore the equations of a group of surfaces which admit of continuous deformation with preservation of conjugate lines.

In the special case where $m = 0$, equations (8) reduce to

$$(10) \quad x_0 = \frac{-2\cos v}{u+v}, \quad y_0 = \frac{-2\sin v}{u+v}, \quad iz_0 = \frac{2}{u+v} - v,$$

which are the equations of a ruled surface whose generatrices are straight lines of zero length. The equations of the corresponding associated surfaces are

$$(11) \begin{cases} x = (U'' + V'')\cos v - (U' - V' + (u+v)U''), \\ y = (U'' + V'')\sin v + (U' - V' - (u+v)U''), \\ -iz = U'' + V'' + U + V - (u+v)U' + \dfrac{(u+v)^2}{2}U'''. \end{cases}$$

The following relations exist between the symbols of Christoffel formed for any surface S_0 and its associate surface S and the Gauss sphere, provided S_0 is referred to its asymptotic lines and S is referred to conjugate lines : *

$$(12)\quad \begin{aligned} &\left\{ {1\;2 \atop 1} \right\}_{S_0} = -\left\{ {1\;2 \atop 1} \right\}' = \frac{D''}{D}\left\{ {1\;1 \atop 2} \right\}_S, \quad \left\{ {1\;2 \atop 2} \right\}_{S_0} = -\left\{ {1\;2 \atop 2} \right\}' = \frac{D}{D''}\left\{ {2\;2 \atop 1} \right\}_S, \\ &\left\{ {2\;2 \atop 1} \right\}_{S_0} = -\left\{ {2\;2 \atop 1} \right\}' = \frac{D''}{D}\left\{ {1\;2 \atop 2} \right\}_S, \quad \left\{ {1\;1 \atop 2} \right\}_{S_0} = -\left\{ {1\;1 \atop 2} \right\}' = \frac{D}{D''}\left\{ {1\;2 \atop 1} \right\}_S. \end{aligned}$$

If we exclude developable surfaces, the second fundamental magnitudes D and D'' of S cannot be identically zero. Now by (1) the fundamental magnitudes of the sphere which corresponds to (10) and (11) are,

$$e = 0, \quad f = \frac{-2}{(u+v)^2}, \quad g = 1.$$

Hence in this case, $\left\{ {1\;2 \atop 1} \right\}' = \left\{ {1\;2 \atop 2} \right\}' = \left\{ {1\;1 \atop 2} \right\}' = 0$. Thus taking account of (1) and (12), we have for (11), $\left\{ {1\;1 \atop 2} \right\} = 0$ and $E \neq 0$. Hence the lines $v =$ const. on (11) are geodesics. Also $\left\{ {2\;2 \atop 1} \right\} = 0$ and $G = 0$ and hence the lines $u =$ const. are minimal lines. Thus the surfaces (11) play a rôle between that of the minimal surfaces, on which the invariant conjugate system consists of two families of minimal lines, and the surfaces of Voss, on which the invariant conjugate system consists of two families of geodesics.

If the general solution θ of an equation of the form $\partial^2\theta/\partial u \partial v = M\theta$ is known, where M is an arbitrary function of u and v, and one can find three particular solutions, ξ, η, ζ, linearly independent of θ, the equations of a surface $S_0(x_0, y_0, z_0)$ referred to its asymptotic lines can be found by six quadratures from the formulas †

$$(13)\quad \begin{cases} \dfrac{\partial x_0}{\partial u} = \zeta\dfrac{\partial \eta}{\partial u} - \eta\dfrac{\partial \zeta}{\partial u}, & \dfrac{\partial x_0}{\partial v} = \eta\dfrac{\partial \zeta}{\partial v} - \zeta\dfrac{\partial \eta}{\partial v}, \\[2ex] \dfrac{\partial y_0}{\partial u} = \xi\dfrac{\partial \zeta}{\partial u} - \zeta\dfrac{\partial \xi}{\partial u}, & \dfrac{\partial y_0}{\partial v} = \zeta\dfrac{\partial \xi}{\partial v} - \xi\dfrac{\partial \zeta}{\partial v}, \\[2ex] \dfrac{\partial z_0}{\partial u} = \eta\dfrac{\partial \xi}{\partial u} - \xi\dfrac{\partial \eta}{\partial u}, & \dfrac{\partial z_0}{\partial v} = \xi\dfrac{\partial \eta}{\partial v} - \eta\dfrac{\partial \xi}{\partial v}, \end{cases}$$

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* Bianchi-Lukat, *l. c.*, pp. 127, 135.

† Bianchi Lukat, *l. c.*, p 132 ff.

where $K = -1/\rho^2$ is the curvature of S_0 and $\xi = \sqrt{\rho}X$, $\eta = \sqrt{\rho}Y$, $\zeta = \sqrt{\rho}Z$. If ξ, η, ζ are chosen such that $\xi^2 + \eta^2 + \zeta^2 = [\phi(u) + \chi(v)]^2$, the associate of such a surface will admit of continuous deformation with preservation of conjugate lines.

As an example, consider the equation $\partial^2\theta/\partial u\partial v = 0$, of which the general solution is $\theta = U + V$. As three particular solutions, assume $\xi = v$, $\eta = \psi(v)$, $\zeta = u$, where $\psi(v)$ is arbitrary. Then $\xi^2 + \eta^2 + \zeta^2 = u^2 + v^2 + \psi^2$. Formulas (13) give for S_0 the right conoid,

$$x_0 = u\psi, \quad y_0 = uv, \quad z_0 = \int(\psi - v\psi')dv. \tag{14}$$

Then for the group of associated surfaces we have from (9),

$$x = \frac{\psi'(uU' - U) + \psi V' - \psi' V}{\psi - v\psi'}, \tag{15}$$

$$y = \frac{uU' - U + vV' - V}{\psi - v\psi'}, \quad z = U',$$

where the primes denote partial derivatives. These surfaces were first investigated by Mlodzieowski.*

For the fundamental magnitudes we find

$$E = \frac{U''^2}{(\psi - v\psi')^2}[u(1 + \psi'^2) + (\psi - v\psi')^2],$$

$$F = \frac{(v + \psi\psi')uU''}{(\psi - v\psi')^3}[V''(\psi - v\psi') + \psi''(uU' - U + vV' - V)],$$

$$G = \frac{\psi^2 + v^2}{(\psi - v\psi')^4}[V''(\psi - v\psi') + \psi''(uU' - U + vV' - V)],$$

$$D = \frac{U''}{\sqrt{u^2 + v^2 + \psi^2}}, \qquad D' = 0,$$

$$D'' = \frac{V''(\psi - v\psi') + \psi''(uU' - U + vV' - V)}{(\psi - v\psi')\sqrt{u^2 + v^2 + \psi^2}}.$$

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* See Goursat, *Amer. Jour. of Math.*, vol. 14, pp. 1–8.

If in (14) we choose $\psi = 1$ we have the paraboloid, and from (15) as its associated surfaces

$$x = V'; \qquad y = uU' - U + vV' - V, \qquad z = U',$$

which are the equations of surfaces of translation whose generating lines are plane and lie in perpendicular planes.

Putting $\psi = \sqrt{1 - v^2}$ and changing v to $\sin v'$, we have from (14), omitting the primes, the helicoid

$$x_0 = u \cos v, \quad y_0 = u \sin v, \quad z_0 = v$$

and, for its associated surfaces, the moulure surfaces

$$x = -(vU' - U) - V' \cos v + V \sin v,$$
$$y = uU' - U - V' \sin v - V \cos v,$$
$$z = U'.$$

Finally, putting $U = c\sqrt{1 + u^2}$, $V = 0$, $\psi = -1/b\sqrt{c^2 - a^2v^2}$, where a, b, c, are constants, and changing u to $\cot u'$, we have from (14), omitting the primes,

$$x_0 = -c/b \cot u \cos v, \quad y_0 = c/a \cot u \sin v, \quad z_0 = -c^2v/ab$$

and for the associated surfaces, the quadrics

$$x = a \sin u \sin v, \quad y = b \sin u \cos v, \quad z = c \cos u.$$

If S is any surface which admits of continuous deformation with preservation of a conjugate system, the different forms which S assumes during the deformation may be regarded as distinct surfaces, so that the deformation leads to a continuous system of applicable surfaces. To find the individual surfaces of such a system, let E, F, G, D, 0, D'' be the first and second fundamental magnitudes of S referred to a conjugate system of lines: then the corresponding magnitudes of any other surface of the system are E, F, G, λD, 0, D''/λ, where λ satisfies the equation *

$$(16) \qquad \frac{\partial\lambda}{\partial u} = \left\{ {12 \atop 2} \right\}'(\lambda - \lambda^3), \quad \frac{\partial\lambda}{\partial v} = \left\{ {12 \atop 1} \right\}'\left(\frac{1}{\lambda} - \lambda\right).$$

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* Bianchi-Lukat, l. c., p. 336.

If the surface S_0 to which S is associated is referred to asymptotic lines, S and S_0 must have the same spherical representation and the symbols $\left\{ {1\,2 \atop 1} \right\}'$ and $\left\{ {1\,2 \atop 2} \right\}'$ will be the same functions of u, v whether calculated from the equations of S or from those of S_0. Using S_0, Codazzi's equations may be written in the form

$$(17) \qquad \left\{ {1\,2 \atop 1} \right\}' = -\tfrac{1}{2}\frac{\partial}{\partial u}(\log \rho), \qquad \left\{ {1\,2 \atop 2} \right\}' = -\tfrac{1}{2}\frac{\partial}{\partial v}(\log \rho),$$

where

$$\rho = \frac{1}{\sqrt{-K_0}} = \phi(u) + \chi(v).$$

From (17) we have for (16),

$$\frac{\partial}{\partial u}\left[\log\left(\frac{\lambda}{1-\lambda^2}\right)\right] = -\frac{\partial}{\partial u}(\log \rho),$$

$$\frac{\partial}{\partial v}[\log(1-\lambda^2)] = \frac{\partial}{\partial v}(\log \rho).$$

Solving these equations for λ we find

$$\lambda = \sqrt{\frac{c+\chi(v)}{c-\phi(u)}},$$

where c is a constant. Then by Bonnet's theorem, to every value of c there corresponds a surface whose fundamental magnitudes are $E, F, G, \lambda D, 0, D''/\lambda$ and the determination of whose cartesian coordinates depends upon the integration of an equation of Riccati. In the case of the surfaces defined by (16) we have,

$$\lambda = \sqrt{\frac{c+v^2+\psi^2}{c-u^2}}.$$

PURDUE UNIVERSITY,
November, 1905.

THE NEW CALCULUS OF VARIATIONS.

THE title above will indicate the intention of the writer to allow himself more freedom of comment in the present paper than is usual in a review. In attempting to form a just estimate of Kneser's Lehrbuch* and in perusing Bolza's Lectures † as well as in the preparation of another paper ‡ dealing with the calculus of variations, the several books and memoirs which I shall mention have necessarily come to my attention, even though some of them have not been used extensively. It appears to me that a short paper which shall give an idea of the various books now before the public would serve two purposes: the valuable one of giving a comparative view of all, and the convenient one of condensing into one a number of separate reviews which might eventually weary the reader.

Of the old calculus of variations the mathematical public knows well. A book which will undoubtedly stand for all time as the last exposition of that theory in what was its best form is Pascal's Calcolo delle variazioni,§ which was published in German translation by Teubner in 1899.|| That this translation — which is the edition to which we shall constantly refer — was thought worthy of publication and was actually the best extant treatise in 1899 is a curious commentary upon the suddenness with which the modern theory leaped into the public arena and upon the secrecy in which the previous developments of that theory had been veiled, especially when we note that the very next year is the date of Kneser's now famous Lehrbuch. To be sure Zermelo had in his thesis given the essence of the Weierstrass theory, and the papers by Zermelo and Kneser seem to have been familiar at least to the translator (see page 65 and footnote, page 65). But if evidence were needed that Weierstrass's theory was not generally known or that the papers mentioned had failed to make a noticeable impression upon the general mathematical public, one need not go beyond the present book in search of it, for the influence of the Weier-

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* Lehrbuch der Variationsrechnung, by A. Kneser, Braunschweig, Vieweg, 1900, 8vo, 306 pp.

† Lectures on the Calculus of Variations, by O. Bolza. See BULLETIN, vol. 12, No. 2 (November, 1905) pp. 80–90.

‡ Article on the Calculus of Variations, Encyclopedia Americana, 1905.

§ Calcolo delle variazioni, by E. Pascal, Milan, Hoepli, 1897.

|| Variationsrechnung, by E. Pascal, translated into German by A. Schepp, Leipzig, Teubner, 1899, 8vo, 146 pp.

strass theory upon this work is at most meagre. Pascal intended to make his book largely historical and he succeeded. This success now saves it from the oblivion which is the fate of many another book of scientifically the same class. The fourteen pages of historical matter at the beginning give only a partial idea of the amount of useful information of that nature in the book, for every chapter contains thorough references and some of the chapters are little else than abstracts of the theory and historical comment.

I shall not enter upon a detailed discussion of the contents, since the problems treated and the methods used are generally well known. Suffice it to say that nearly every problem of the old theory is mentioned in its general form: the simplest problem, the extensions to cases where higher derivatives enter the integrand, where multiple integrals occur, where auxiliary conditions are imposed, and so on. Thus most of the formal results are presented here in compact and convenient form. But the critical spirit is not so marked as it must be henceforth. As a single example may be mentioned the absence of exact statements concerning the very definition of a minimizing curve. Indeed the author points out (pages 16 and 17) that "certain categories of curves" are arbitrarily excluded as comparison curves on account of the fact that the nature of the "variations" allowed restricts the comparison curve so that "all the derivatives approach the derivatives of like order for the supposed solution." But this remark goes no further than did the almost identical remark made by Legendre (see § 30, page 109). Surely Zermelo's dissertation should have been mentioned in this connection, especially in the discussion of Newton's problem (§ 30),* and the failure to take account of Zermelo's exposition of Weierstrass's theory results, both here and elsewhere, in the well-known misstatements common to all the older works.

Finally, one feature which remains valuable should be mentioned. Practically the whole of the last thirty pages — nearly one-fourth of the book — is devoted to special problems and applications, including most of the famous problems and such general theories as those of minimum surfaces, geodetic lines, etc. While the treatments given do suffer from the lack of precision noted above, it should be kept in mind that the solutions obtained must include the rigorous solutions at any rate, and that the curves which are declared to be minimizing curves would in general at least render the integral a "weak" minimum. In seeking a rigorous solution, therefore, these results are still by no means to be despised.

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* Compare Kneser, Encyklopädie d. Math. Wiss., II A 8, p. 609.

A very useful bibliography of the subject, which practically exhausts the literature up to 1890 at least, forms a fitting sequel to a book which Pascal has made curiously important from a historical standpoint.

In striking contrast to the spirit as well as to the contents of Pascal's book, there appeared in 1900 a new treatise by Kneser.* I have elsewhere † claimed that this book opened the doors of modern research in the subject to the general mathematical public, and its importance really merits a much more extended and probably a more favorable review than I shall be able to make at this time. The book would undoubtedly have been reviewed in these pages ere now, had the task seemed entirely easy and attractive to any one of several Americans who were otherwise interested in its appearance and its contents.

It is especially easy to draw broad comparisons between Kneser and Pascal. The latter's treatment is very clear, it gives a good general view of the subject as Pascal knew it; its theorems and other statements are well outlined and set in prominent types, it leans heavily toward historical comment; but it is lacking in rigor from the modern standpoint. Kneser's work, by contrast, is not lucid, its arrangement gives only a clouded view of the author's own conceptions and of the subject itself, the theorems and other similar statements are hidden amid a mass of discussion as if with conscious and consummate cunning, it shows an apparent tendency to conceal historical development, but it is a vast advance over any former work in its exactness.

Kneser published at the same time an article in the Encyklopädie,‡ which has been sent out only this year, but which was written and set up along with the Lehrbuch, and which should be considered at the same time. I shall refer to this second treatment as Kneser's *article*.

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*See first footnote, p. 172.

†See BULLETIN, loc cit.

‡ Encyklopädie der Math. Wiss., II A 8., pp. 571–625. Article entitled Variationsrechnung.

In considering these two treatments by Kneser it should be noted that both of them formally appear in print under the date 1900, though the article has not been given to the public until very recently. It follows that we shall not expect to find in either of them any of that fundamental work which Hilbert has been doing since 1900. It will be very useful in this connection to refer to Bolza's Lectures on the calculus of variations, which has been reviewed in the BULLETIN,* for Bolza in his chapters numbered V and VI follows Kneser rather closely. On account of the greater clearness of style, and on account of the development of the theory in the interval between the two dates of publication, Bolza's presentation is an enlightening commentary upon Kneser's work — an almost indispensable aid in deciphering much that is obscure in Kneser's own point of view. Surely a review of Kneser's Lehrbuch is immensely simplified through the existence of Bolza's book.

The first striking characteristic of Kneser's Lehrbuch is his exclusive use of the parameter representation of Weierstrass. The standpoint assumed by Bolza in his Lectures † will appeal to most teachers on account of its pedagogical correctness. Thus Bolza apparently appreciates the parameter representation quite as fully as does Kneser, but he sees and utilizes the possibility of giving the main essentials of the theory in the simpler asymmetric form first, and then passing to the more involved parameter representation later, by means of generalizations which are often trivially simple when the main facts are already established. This advantage in style Kneser missed: in consequence his work suffers in clearness, though it might be possible to write a very readable book based on the parameter representation alone. Aside from this, however, I am inclined to believe with Bolza (Lectures, page 115) that it is unjustifiable to discard the asymmetrical forms entirely in favor of the parameter representation.

Let us pass in review briefly the various chapters. The first chapter contains a statement of the problem in parameter form and the formal transformation of the first variation. The theorems of Weierstrass concerning integrals which are independent of the choice of parameter, and the corresponding methods of transforming the first variation were previously known only through the meagre publications of Weierstrass's students, and form a necessary and important part of the introduction. The assumption on page 10 that the second derivatives exist for the solution is now known to be unnecessary, through the DuBois-Reymond-Hilbert proof, but such an assumption does not vitally injure the work and might even be defended as justifiable by one who was conscious of its superfluity.* The non-parameter form is explained in § 5 as a special or dependent case. Kneser's article in the Encyklopädie shows a striking contrast to the treatment of the Lehrbuch in that the parameter representation is mentioned in the article only in the briefest possible manner. The difference may be due to the great limitation which the author imposes upon himself in the use of work due to Weierstrass in the article (see Encyklopädie, II, A, 8, page 608, footnote 121*a*). In the Lehrbuch, though Kneser limits himself to work actually published by others, he does at least permit himself enough freedom to use Weierstrass's essential methods as presented by Zermelo and others. Again, the article is historical to a large extent, so that Weierstrass's results consume relatively little space, and the impossibility of giving an account of earlier investigations in parameter form may well account for the extreme difference between the Lehrbuch and the article.

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* Vol. 12, No. 2 (November, 1905), pp. 80–90.

† See second footnote, p. 172.

The second chapter of the Lehrbuch (pages 10–43) is devoted to the derivation of necessary conditions. The proof of Euler's (or Lagrange's) condition follows easily from the transformed first variation by means of DuBois-Reymond's lemma (§8). In this connection, Kneser dwells upon the now familiar idea, which is due to Kneser himself, of the extremals, i. e., the solutions of Euler's equation. After giving a list of examples, Kneser proceeds in § 10 to what is perhaps the key note of his whole theory:† the consideration of integrals in which the end points are variable. As a matter of fact, these problems lead naturally to a generalization of the theorems concerning geodetic lines on a surface, especially after the introduction of the important idea of transversal curves. The whole problem of variable end points may also be thought of as an analogon of the boundary value problems of the theory of differential equations; it is perhaps even more interesting from this point of view than from the analogy to geodetics, but both points of view should be kept in mind and both might have been emphasized even more than Kneser has seen fit to do, without overstepping the proper bounds of the calculus of variations. A brief consideration of the isoperimetric problem closes the chapter.

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* Compare, for example, the results in § 17, p. 58.

† See Bolza, Lectures, Chapter V.

One expects to find next a treatment of the so-called "second variation," according to the methods of Jacobi and his school. This expectation is based wholly on tradition and on the stress which Kneser lays in his article on the same methods. Here, however, Kneser leaves the beaten paths absolutely and begins at once a consideration of the sufficient conditions. The ideas developed are new and important in almost every instance. The idea of a field of extremals (§14) is the first and most fundamental. The following article on the extension of Gauss's theorems concerning geodetics is scarcely less important. Then follow the essential methods according to Kneser's theory for the proof of the sufficient conditions (§§ 16, 17), including the introduction of Kneser's curvilinear coordinates based upon a set of extremals and a corresponding set of transversals, and the essential distinction between strong and weak minima. These articles lead up to sufficient conditions, which Kneser sums up on page 60 under the titles "Jacobi's condition" and "Legendre's condition." The former is essentially the same condition as that elsewhere given as Jacobi's condition. The condition called "Legendre's" is that known to Legendre only in the case of weak minima. In the case of strong minima the condition is radically different, since it practically involves "Weierstrass's" condition, and might well be called "Weierstrass's" condition rather than "Legendre's." I shall not discuss in detail the remainder of this chapter, though it is important throughout. Among the topics considered are the methods of Weierstrass, including the E function; the envelope of a field of extremals; the Jacobi-Hamilton theorems. An amount of this matter will be found either reproduced or independently collected in Bolza's Lectures. A few errors or oversights in this chapter have been noted by Bolza in his Lectures and by Zermelo and Hahn in the Encyklopädie. Thus Kneser's proof of the existence of a field (§ 14) must be supplemented by a theorem due to Osgood (see Bolza, Lectures, page 176); and the statements of Jacobi's condition require certain revision (see Encyklopädie II, pages 630, 632), which have been given by Osgood, Bolza, and Bliss in memoirs quoted by Zermelo and

Hahn. The latter revisions have been conducted by means of considerations of the so-called "second variation," which would seem to indicate a present necessity for retaining the older considerations of Jacobi, at least until the proofs in question have been made upon the basis of the newer theories.

Kneser's fourth chapter treats the isoperimetric problem both for fixed and for variable end points. The principal methods and results of the chapter are presented in somewhat simpler form by Bolza (Lectures, Chapter VI). While the theorems are frequently essentially the same as those previously stated, some of the results and some of the methods employed are new. Any future work on this subject will necessarily use many of the ideas which Kneser introduces in this chapter.

The last four chapters are devoted to discontinuous solutions, the problem in which higher derivatives enter in the integrand, the most general problem of the calculus of variations, and double integrals, respectively. These chapters are certainly important, and they make the work complete in a sense in which it could not be without them. The previous chapters give the character to the book, however, and they are the especially interesting and important portions. Such an explanation is necessary in passing them over with no detailed mention, in order to avoid a misconception on the part of the reader. The high plane upon which Kneser has placed his work in the earlier chapters is fully maintained to the end.

In general, Kneser's Lehrbuch must surely be assigned a very high scientific value, and its many contributions will surely remain essential for the further development of the subject for many years. It is only from the pedagogic side that essential criticism can justly be made. The work is of monumental importance; it might have been of truly immense influence upon all classes of students of mathematics, had it not been for the unfortunate style of the author and for the resulting unnecessary complication and intricacy of statement and proof which characterize it from the standpoint of presentation. A previous remark of mine * to the same effect has been misunderstood in a curious manner by Professor Haussner (*Fortschritte der Mathematik*, volume 33, page 379), who himself repeats my own criticism and my own praise in slightly greater detail. For this reason I might now insist that a presentation can justly be criticized for its complication when and only when it is unnecessarily intricate, i. e., in so far as the complication arises from the style of the writer rather than from the nature of the problem. And can there be any doubt that any intrinsic difficulty or intricacy in the subject matter itself calls for especial care upon the part of a writer who wishes adequately to present the subject or any portion of it?

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* BULLETIN, vol. 9, No. 1, October, 1902, p. 24.

Kneser's article in the Encyklopädie, which has been quoted above, is scarcely to be classed with his Lehrbuch. Its whole tone is rather historical than scientific. Its value is further limited by the attitude of the author toward Weierstrass's work, which, though well known through practically public lecture notes, is refused recognition here except when accidentally published by another author. The scruple against presenting the unpublished work of another man is honorable in general, but in this instance no possible harm could have been done, and the fame of Weierstrass would be better guarded by a frank reference to his unpublished work than by silence at a time when a large part of the modern developments in the calculus of variations can be traced directly or indirectly to his influence. The references given in footnotes supplement to a very considerable degree the lack of such references in the Lehrbuch; otherwise the Lehrbuch is a more systematic presentation of the subject and is, of course, much more detailed in its treatment than the article. It would therefore seem that the only present value of the article is its historical value, including its copious notes.

The article by Zermelo and Hahn,* which immediately follows Kneser's article in the Encyklopädie, is much more important at the present moment, since the work reviewed in it is not as yet fully reproduced in any treatise. To be sure a number of the results are given by Bolza in his Lectures, but there are very many points of interest upon which Bolza does not touch. This article covers the period from 1900 (when Kneser's article was finished) to 1904 (when both articles actually appeared). It is probable that no equal period of time has seen such an advancement in the subject as this, and it is certain that at no time have so many minds been devoted to its investigation. The article covers only fifteen pages and, therefore, it is necessarily only a summary of facts. The footnotes, however, contain complete references to the original sources. The recent work of the Hilbert school seems to predominate in the exposition in the constant recurrence of the important ideas of the Hilbert invariant integral and the Hilbert a priori existence proof. Of course many authors in no way allied with Hilbert are repeatedly mentioned. In particular the work of Weierstrass was an essential precursor of that of Hilbert, and Kneser's work is of great importance, independently of any of Hilbert's work. The next most prominent names seem to be those of Osgood, Bolza, and Bliss; these three Americans and to a lesser extent a few other Americans have not only made the modern calculus of variations familiar to the American public, they have advanced to a marked degree the general knowledge of the subject in the important papers referred to in this article. The topics are the Weierstrass theory, the simplest problem, the isoperimetric problem, the general problem, applications, and the existence proofs. In many particulars this article is at present the most authoritative and exhaustive presentation of the results of recent investigations in the subject.

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* Encyklopädie der Math. Wiss. II A 8a, article entitled "Weiterentwickelung der Variationsrechnung in den letzten Jahren," pp. 626–641.

The last book I shall mention is also of American origin.* It represents substantially the lectures delivered in regular courses on the calculus of variations by the author, Professor Harris Hancock, at the University of Cincinnati. Professor Hancock bases his work almost exclusively upon the lectures of Weierstrass and Schwarz which he heard at Berlin, and consciously disregards the developments of later years. Since the work of Weierstrass is already fairly well known in even more authoritative form through the well-known lecture notes at Berlin, the present book is shorn of much of the value it might otherwise have from a historical standpoint. An official publication of Weierstrass's work by those who now hold his original papers would possess extreme interest in showing precisely Weierstrass's point of view, whereas it is not always evident in the present book whether the statements and the proofs, in the precise form in which they are given, are due to Weierstrass or to Hancock. For this reason it is difficult and rather unnecessary to review the book in detail. It may be noted that a considerable portion of the contents has already been published by Professor Hancock in the *Annals of Mathematics* (first series, volumes 9–12).

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* Lectures on the Calculus of Variations, by H. Hancock, Cincinnati, University Press, 1904, 8vo, 292 pp.

Beside the books and articles mentioned in this review, several of the recent treatises on the calculus contain a chapter on the subject. Among these should be mentioned at least the new edition of Serret-Bohlmann, Differential and Integralrechung, volume 3, and Goursat, Cours d'Analyse, volume 2. These books will doubtless be reviewed in their entirety in the BULLETIN, and I therefore satisfy myself here with a mere mention.

The works mentioned attest in the strongest possible manner the extraordinary vogue into which the calculus of variations has suddenly sprung. The cause is not far to seek: it is the revelation through the work of Weierstrass, Kneser, Hilbert and others that the calculus of variations is susceptible of the same exquisite rigor which had previously existed only in the theory of functions of a real variable, and that a wide field of research and rich discovery was opened by such methods.

Although the end of these investigations has by no means been reached in this single subject, it is not premature to suggest the analogous development of other mathematical theories along equally rigorous lines, and also the construction of a supplementary theory in each of them which shall be as rigorously applicable to general geometric problems as is the Weierstrass theory in the calculus of variations. E. R. HEDRICK.

COLUMBIA, MO.,
November, 1905.

GRANVILLE'S DIFFERENTIAL AND INTEGRAL CALCULUS.

Elements of the Differential and Integral Calculus. By W. A. GRANVILLE, Ph.D., with the editorial cooperation of PERCY F. SMITH, Ph.D. Ginn & Co., 1904. 463 pp.

So many text-books have been written upon the elementary branches of college mathematics that a raison d'etre can properly be asked upon the appearance of each new work. The great number of American text-books upon such subjects as college algebra, trigonometry and calculus, duplicating one another in aim and character, is in striking contrast with the paucity of our text-books upon more advanced mathematical subjects. What, then, we naturally ask, is the purpose of this new treatise, and what does it seek to accomplish which has not been already accomplished?

In the introduction the authors tell us that "the present volume is the result of an effort to write a *modern* text-book on the calculus which shall be essentially a drill book." In making this quotation I have taken the liberty to italicize the word "modern," inasmuch as it seems to me that the chief interest of the book, and the first as well as the last questions concerning it center in this word. The need of modern text-books on calculus is recognized by every one who is familiar with its modern developments and reconstruction. The older English text-books — as, for example, those of Todhunter and Williamson — fail altogether to satisfy the recent demands for precision and rigor. Some of the most fundamental concepts have not been formulated with mathematical definiteness, and hence the consequences of concepts and definitions can not be clearly traced. As a result, the theorems are commonly stated without restrictions. But, as universals, they are rarely true. Thus while the student gains a good knowledge of rules and formulas, he is left in ignorance both of the limits of their applicability and of the restrictions which must be imposed upon the functions to bring them under the theorems. Text-books of this nature are only too likely to produce thinkers of the same indefinite character; for like breeds like. Recently two or three English text-books, in particular Lamb's and Gibson's, have been written to remedy this defect. But notwithstanding manifold excellences these books are scarcely serviceable for a first course in our American colleges.

The composition of our classes often gives rise to grave pedagogical difficulties in teaching the calculus. Very rarely do these classes consist of a few choice spirits, for whom calculus is a preparation for the higher mathematics. The bulk of the class usually consists of men of very varied aims and ability. Some there will almost certainly be who are pursuing calculus for the sake of its applications in physics, chemistry and other sciences, as for example the engineer, by whom calculus is often used as a mechanical tool; some there may be who are preparing to teach mathematics in secondary schools, who do not care to advance into its higher branches and yet wish to gain a glimpse of modern methods and conceptions; others perchance who finish their mathematical education with calculus and aim merely to review and strengthen their previous knowledge; others still, who, under our elective system, take the course by instinct, without any definite object, because they have "a fondness for examples" and for working out the neat riddles and puzzles which mathematics affords. Such elements, and even others, may be included in a class. In such a constituency we find a "mathematisches Publicum" of the greatest interest and importance, from which we must continue to draw much of our best mathematical talent. It is, moreover, this Publicum which must be carefully cultivated to preserve for mathematics that wide influence on thought which it has had in the past, and which is so admirably traced in Merz's History of European thought in the nineteenth century.*

How shall calculus be taught to classes of such varied composition as has been described? How far is it possible, and, if possible, is it desirable and wise to introduce the modern developments into elementary teaching? The position assumed by the authors in their new treatise in answer to this question seems to the reviewer both a sane and a practical one.

First of all, it must be kept in mind that for an ordinary class the text-book must be primarily a drill book. Dr. Granville understands thoroughly this need, and it may be confidently predicted that his text-book will be a most decided success in the class room. Unusually attractive in appearance, the volume presents the subject matter clearly and bears the marks of a teacher who feels a student's difficulties. There is an unusually large and well-graded stock of examples, from which the student will gain an admirable review in the branches of mathematics preceding calculus. In this particular Granville's book resembles Osborne's treatise. It is provided also with good collections of formulas, of integrals, and of curves, for reference.

How to write such a drill book and keep it in touch with modern analysis is the chief problem of the author. This accord has been sought by drawing upon intuition for certain principles, of which the proof is postponed to a second course in the calculus. Upon these principles as a firm foundation the calculus can then be securely built. Such, at least, is my interpretation of what has been and should be sought.

Consider first the application of this method to the differential calculus. Here the foundations for a rigorous treatment are Rolle's theorem and the theorems of mean value. These are easily approached from the intuitional side. No attempt is made by the author to reproduce the delicate analysis by which they are established. He clearly recognizes that such an analysis is not suited for an elementary course. But his very brief intuitional and geometric treatment will be grasped at once by the student. Once obtained, these theorems serve as a sound basis for the theory of indeterminate forms, for Taylor's formula * and series, for the theory of envelopes, etc. This plan on the one hand has the advantage of avoiding considerations which are premature, and on the other hand removes the necessity of unlearning subsequently false demonstrations and methods. No such fallacious proofs need be foisted upon the learner as are found in many text-books for the expansion of a function into Taylor's series or for the theorem that in the partial differentiation of a function of two variables the order of the differentiations is immaterial. Such proofs confuse and enfeeble the perceptive powers of the learner.

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* This book is pervaded with a mathematical spirit. The influence of mathematics upon thought is considered, beginning with the seventeenth century.

Complete success in infusing an elementary text-book with the modern spirit could scarcely be expected. Let me point out one or two places where in pouring the new wine into the old bottles the skin has been badly cracked. One of these rents is found in the very theorem which was mentioned last; namely, that in obtaining $\partial^2 f(x,y)/\partial x \partial y$ the order of the two differentiations may be reversed. Here, in applying the mean value theorem to the two increments

$$f(x+\Delta x, y) - f(x, y) = \Delta x \cdot f'_x(x+\theta_1 \Delta x, y) \qquad (0 < \theta < 1),$$
$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) = \Delta x \cdot f'_x(x+\theta_1 \Delta x, y+\Delta y),$$

the fatal slip is made that a common value θ_1 is taken for θ in the two equations. (Cf. equation C, page 206.) The oversight contained in the proof (§ 124) of the rule for evaluating the indeterminate form ∞/∞ is certainly surprising.†

Let us turn now to the integral calculus. Here the first pedagogical difficulty relates to the definite integral

$$\int_a^b \phi(x) dx,$$

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* Called by Granville "the extended theorem of mean value" (§ 118).

† When h approaches 0, $f(b)$ approaches ∞ so that $f(b)/F(x)$ itself tends to take the form ∞/∞ !

defined of course as the limit of

$$\sum_{x=a}^{x=b} \phi(x)dx.$$

Shall the existence of Darboux's upper and lower integrals be first demonstrated, and afterward their equality for a continuous function? Or shall the scope of the concept of an integral be contracted by taking all the increments dx equal to each other and multiplying each increment by the value of $\phi(x)$ at the beginning of the increments, as is done in most of our textbooks? Granville and Smith wisely seek a middle course. Neither Darboux's integrals are introduced nor the above restricted concept of an integral. Appeal is again made to geometric intuition. By the consideration of the area under the arc of a continuous curve it is made clear that the two restrictions above described are unnecessary, and a corresponding analytic investigation is added, though of a somewhat incomplete character. I say "incomplete" because the investigation proves only that if $\phi(x)$ is continuous between $x = a$ and $x = b$, and *if there is a corresponding function* $f(x)$, *of which* $\phi(x)$ *is the derivative*, then there must be a limit for

$$\sum_{x=a}^{x=b} \phi(x)dx.$$

This is shown by the aid of the mean value theorem. Evidently the authors have thought it wisest to content themselves with this result. But in the opinion of the reviewer the all important fact should be blazed before the student that every function, continuous between a and b inclusive, has an integral between these limits. This theorem is not even stated, although obviously implied in the concept of the area under a curve. Moreover, it is needed by the author himself to complete his derivation (page 379) of the formula

$$s = \int_a^b \sqrt{1 + (f(x))^2}\, dx$$

for the length of a curve.* A suitable and simple proof of the theorem could be given readily, without entering upon all of the work necessary to establish Darboux's upper and lower integrals, with the aid of the geometric intuition that when $y = \phi(x)$ is continuous in an interval, a value δ can be found so small that the variation of $\phi(x)$ in any subinterval less than δ in length will be less than any arbitrarily assigned positive magnitude.[*] This very fundamental intuition is almost hidden by the author in the fine print of page 370; it should be thrown into bold relief.

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* The theorem could have been used also as the basis of other formulas of the integral calculus, and is then preceded by a summation. This is the integral method of proof. The author chooses the differential method. Cf. §§ 223, 227, 228.

A second deficiency is the failure to exhibit the two definitions of the definite integral — obtained (1) from the limit of a sum and (2) from the difference of two values of the indefinite integral — as two wholly independent and by no means coextensive definitions.

In these and some other particulars it is to be hoped that when a second edition of the book appears, it will be permeated yet further with the modern spirit. The restrictions respecting continuity imposed upon the derivatives in various theorems might be incorporated advantageously into the theorems themselves, instead of being relegated to footnotes or altogether omitted (e. g., § 118). Continued insistence that the student shall trace the effect of so primitive a concept as continuity and shall note the conditions introduced successively into a theorem or into a chain of theorems will help to impress upon him the fact, too often overlooked, that careful observance of the limitations imposed by the reasoning is fundamental in algebraic analysis as well as in synthetic geometry.

I have dwelt, perhaps unduly, upon certain aspects of an admirable text-book because of my great interest in the adaptation of calculus to the beginner and because of a belief that the most practicable plan is that followed by the authors, to use intuition when and only when it has been justified by exact analysis and then to build boldly upon it.

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[*] For first, it could be shown immediately from this intuition that the difference between the values of

$$\sum_{x=a}^{x=b}\phi(x)dx$$

for any two partitions whatsoever of the interval (ab) can be made as small as we choose by taking the maximum of $|dx|$ sufficiently small. Then it would remain only to prove that for some system of partitions $\Sigma\phi(x)dx$ converges to a limit. The latter fact is evident since for any infinity of values of $\Sigma\phi(x)dx$ there must be at least one point of condensation.

A few words should be added about the contents of the book. The usual range of topics is included, without neglecting applications in mechanics and with the addition of brief but suitable chapters upon ordinary differential equations and upon tangent lines and planes in space. Special attention is given to the parametric representation of curves. An unfortunate omission will be noted under the topic of differentials. It is not proved nor even remarked that the differential and increment of a function $f(x)$ differ from each other by an infinitesimal of higher order, although the proof of this important fact would occupy only a few lines. The application of differentials to the approximate computation of small increments of $f(x)$ is simultaneously excluded. The corresponding omissions in the case of the differential of a function of two or more variables are especially to be regretted, for the differential (§§ 136, 137) is left devoid of significance when the variables are independent.

In conclusion, generous recognition should be accorded to the care which has been bestowed upon the work. At many points improvements over our current text-books will be noticed, not in themselves sufficiently important to dilate upon but having together great cumulative force. As an instance, I shall cite the inclusion of a real proof that two functions which have a common derivative can differ only by a constant. The introductory chapter on the concepts continuity, function, and limits can also be especially commended, and the chapters on series and the expansion of functions. I know of no work which has greater promise of success in our college classes.

EDWARD B. VAN VLECK.

GENEVA,
September 20, 1905.

THE FOUNDATIONS OF SCIENCE.

Wissenschaft und Hypothese. Von HENRI POINCARÉ. Autorisierte deutsche Ausgabe mit erläuterenden Anmerkungen von F. und L. LINDEMANN. Leipzig, B. G. Teubner, 1904. xvi + 342 pp.

NOT logical enough for the logician, not mathematical enough for the mathematician, not physical enough for the physicist, not psychological enough for the psychologist, nor metaphysical enough for the metaphysician, Poincaré's Science and Hypothesis can hardly give the satisfaction of finality to any one; and yet it probably comes nearer to satisfying the requirements of all these classes of investigators than any single book of our acquaintance. To the author this antithesis ought to be the source of the greatest gratification, in view of his position that a theory is never of greater service to science than when it breaks down. For this little book breaks down constantly, although it is full of brilliant suggestion on every topic. Were it possible to find from the pens of the leading logicians, psychologists, physicists, and metaphysicians such works as this from the pen of a supreme mathematician, there would be at hand the material for reaching a satisfactory and unified theory of science — provided, of course, that such a theory is possible. But is it possible? And even if, to follow Bôcher's happy phrasing,* some method of successive approximation is really leading us toward absolute rigor and unity, why should we assume that we shall ever reach the limit, or even that the limit exists?† May we not be forever forced to rest content with a mere feeling of convergence —slower or faster, and very likely asymptotic at best? If so, works like the present will always remain necessary and highly useful, perhaps more useful than those which give a more consistent theory which is less true to natural limitations.

In Part I the author analyzes the concepts of number and magnitude. He affirms (page 17) that the principle of mathematical induction and it alone can teach us something new. The argument is plausible but seems at times too narrow, founded too much on the principle of arithmetization in its most meager form, and too closely associated with the syllogism as the fundamental method of reasoning. In geometry, if not in algebra, one feels conscious of other methods of proof, other methods of passing from the particular to the general and conversely. Furthermore the symbolic logicians have pointed out the fact that other modes of reasoning than the syllogistic are valid and essential parts of our logical machine. In the subsequent discussion of irrational numbers and the continuum, the author's point of view comes emphatically to the foreground, namely, that it is physics which is the real thing, that our mathematics is purely arbitrary and a matter of convenience alone, always guided however by our sense impressions and hence somewhat limited in its arbitrariness as far as practical results are concerned. He says: "The mind does not use its creative faculty except when experience forces it to do so" (page 30.) Although a pure mathematician, imbued to a less extent than is Poincaré with the idea that it is applied mathematics which is essentially important and that pure mathematics is only relatively so, might dissent from this opinion, there is contained in it at least a highly valuable pedagogic suggestion. The main difficulty in getting students interested in pure mathematics consists in overcoming their inability to see what it is all for.

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* BULLETIN. vol. 11 (1904), p. 120.

† It should be remembered that in order to classify and otherwise deal with experience, science is bound to idealize. For different purposes different idealizations are convenient, and these may sometimes be mutually contradictory without introducing great inconvenience. Take, for instance, the conception of infinite. The idea of finiteness is definite. When, however, we negative the idea, we are often led to harmless contradictory statements. Thus "the portion of the plane which is at infinity is a line" and "the portion of the plane which is at infinity is a point" cannot both be true, but are in constant use as different idealizations, and introduce no difficulty when each is kept in its own field. Recently a great deal of disagreement has arisen in trying to relate the transfinite ordinals and cardinals — perhaps they are harmlessly contradictory.

Part II on space and its relations to geometry and experience is one of the most original and suggestive in the book. A few quotations will serve to exhibit the author's point of view, if not his argument: "The geometric axioms are neither a priori synthetic judgments nor experimental facts. They are merely conventions, or in other words only disguised definitions. Geometry is not an experimental science, though experience guides us in setting up the axioms. The question whether the euclidean geometry is the *true* geometry has no more sense than whether the metric system of measurement is the true system (pages 51, 73, 138)." Bôcher has recently discussed this point of view, and come to conclusions with which we are in substantial agreement.* Poincaré goes on to analyze with care the relations between the geometric space which we imagine on account of its convenience and the various spaces which our senses of sight, touch, and motion afford. The characteristics of homogeneity and isotropy, for instance, are not due to any individual one of these spaces, but arise from studying the laws under which our sense impressions succeed one another. Here the author's point of view is largely that of the psychologist. It may be well to note that the French mathematicians are not content with the purely logical development of geometry from a set of more or less artificial postulates. They insist that psychology be taken into account. Picard writes:* "In these questions, the geometric logicians appear to pay too little attention to psychology, and to the information furnished us by uncivilized peoples." (He is referring primarily to the concept of number, to which the argument applies with certainly no greater force than to geometry.)

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* Loc. cit., p. 124.

This question of how far psychology of individuals or of races should be heeded is very interesting and brings up the whole matter of what the axioms or postulates are for and in how far they actually characterize real space. The mathematician is wont to say that, if the system of postulates is complete, it characterizes completely the (mathematical or logical) space in question. By this he means that he has defined, whether by postulates or nominally, a set of elements connected by certain relations and capable of being put into one to one correspondence with the elements of space in such a way as to leave the relations invariant. (As the elements in question are generally merely the real elements, the correspondence may cease to be one to one if extended to the domain of imaginaries.) For him space is any one of these sets of elements, or if he uses the principle of abstraction it is the class of all such sets. Thus his postulates are both necessary and sufficient for the demonstration of all theorems in his space, and for him the complete story of his space is told. But for the psychologist and the metaphysician the complete story is not told in any so easy fashion. The psychologist is interested in tracing space concepts to their origin in sense experience, the metaphysician, in estimating the objective validity of these concepts. Whether the mathematician shall sometime be able to help them further than at present is difficult to say. Lie, as a geometer, distinguished between groups that were similar; and perhaps the logicians may some day get a sharper criterion than the one to one correspondence.

Force is the title of Part III, in which the subject of mechanics, in its widest significance, is treated. Poincaré avowedly passes over the difficulties of space, which he has just been treating, and of time, of which he had included no discussion in this book, and proceeds to difficulties essentially dynamical with the following characteristic preface (page 92): "Thus absolute space, absolute time, and even geometry are not conditions which impose themselves on mechanics; they are no more prerequisite to mechanics than the French language is logically prerequisite to the truths which are expressed in French. It would be possible to try to formulate the fundamental laws of mechanics in a language independent of all these conventions. Naturally the formulation of these laws would become more complicated, because all these conventions have been imagined precisely for the purpose of abbreviating and simplifying that formulation." It should seem, then, to use the author's guiding idea of convenience, that mechanics is rather a prerequisite to geometry than geometry to mechanics. Thus geometry does become in a certain large sense an experimental science, even if the experiments must be made upon objects which are not geometric but physical. (See the discussion by Bôcher.)

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* BULLETIN, vol. 11 (1905), p. 405.

A little farther on in the text (page 100) is made a statement upon which the metaphysician may well meditate: "That a definition may be of value it is necessary that it shall teach us to *measure* force; this, moreover, is sufficient; it is by no means necessary that the definition teach us what force *as such* is, nor whether it be the cause or the effect of motion." Much misunderstanding and useless discussion between metaphysicians and scientists might be avoided if the former realized that what the latter needs to *know* about force is just what the careless baggage man needs to know about a trunk — its magnitude and its direction — and that what the real content of the thing is may better be left to the metaphysician himself, as a sometimes too officious customs officer, to examine. (The same remark probably holds equally well with regard to the insertion of psychological elements among our geometric axioms.) But who can resist the temptation to examine more critically than is absolutely necessary for the mere physicist? The author himself goes further and comes to the conclusion, similar to that for geometry, that the principles of mechanics are but conventions or disguised definitions, quite arbitrary except in so far as suggested by experience as the most convenient to adopt. It is during this discussion that the now famous formulation of the principle of the conservation of energy occurs, namely: "There is something which remains constant." And another statement: "The two propositions, 'the earth turns' and 'it is more convenient to suppose that the earth turns,' have one and the same meaning." And again, apropos of the law of acceleration: "If the acceleration of a visible body appears to depend on something else than the positions and velocities of other visible bodies and of the invisible molecules whose existence we have previously been brought to admit, nothing prevents us from supposing that this something else is the position or velocity of other molecules whose presence we have hitherto not suspected; and thus the law may be kept."

It requires these rather extensive quotations to bring clearly to light the author's underlying philosophical principle of convenience.* One might almost call his philosophy dogmatic. For, after obtaining a principle of wide application and becoming thoroughly used to interpreting nature or classifying experience by its aid, the most convenient thing to do is to introduce new terms and keep the principle rather than to find a new principle. Now undoubtedly for the mathematician such a procedure is always possible. For the physicist, however, it seems at times very artificial; and while it may serve as a first rate pragmatical doctrine to a certain recent school of metaphysics, there is apparently a decided tinge of superficiality about it. When one considers as practically identical the statements that the earth turns and that it is more convenient to imagine that the earth turns, he takes a position which seems to be either practical agnosticism or pure idealism, according to the interpretation one adopts. Whether this position is interpretable in such a way as to lead to a metaphysics satisfactory from all sides, only time can decide. Certainly, however, it is a great service that Poincaré has done in pointing out so clearly the fact that convenience plays a large rôle, that our knowledge is, as it were, a viscous fluid containing in itself traces of all knowledge that has been, with the weaker and more remote in time ever becoming less effective in determining the present state. The author seems to feel confident that we are approaching a steady state.

The last part of the work deals with nature. Before entering upon this subject the author draws a useful distinction between a law and a principle (page 141): "A law expresses a relation between two real terms A and B; but it is not rigorously true. We introduce arbitrarily an intermediate term C which is more or less fictive and which by *definition* is that which has to A *exactly* the relation expressed by the law. Thus our law is decomposed into an absolute and rigorous principle expressing the relation of A to C and an approximate experimental law expressing the relation of C to B." He goes on to say that in entering upon the study of nature we enter the domain of laws properly so called. From this it is evident that in studying mechanics it is principles which concern us, whereas in studying physics it is laws. It is difficult to see how any such finely drawn distinction can exist in view of the most recent developments of electrodynamic theory. It is becoming more and more convenient to consider electricity as fundamental and mass as derived therefrom. That Poincaré did not foresee this and take it into due account is no blot on his excellent little book; but it goes to show how careful one must be in asserting, for instance, that even the euclidean geometry will always remain the most convenient. Naturally euclidean geometry will always be taught, and so will newtonian mechanics; they will remain the first and simplest approximations to ultimate theories of space and moving matter. But when one is treating the fundamental question of the relation of hypothesis to science, it is not the first but the last available approximation which is of importance. The author would probably be the first to admit this. *

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* Compare Mach's well known principle of economy in thought.

Although what precedes may go some of the way toward giving an impression of what the book contains, it cannot bring out the charm, frankness, and directness of the text. Fortunately one may take his choice of reading in the original French or in the excellent German translation. † If, however, the matter of language is of no very great importance to the reader, he should by all means choose the translation. For the translators have done more than give a faithful rendering of the text. They have written a preface which sets forth clearly the essential points in the argument, and have appended about a hundred pages of notes of great mathematical, historical, and bibliographical value to the reader. Finally they have added an exhaustive index. Why the French so persistently leave the indexes out of their books is hard to imagine. The practice is certainly a great inconvenience, especially in such a work as the present, where constant reference to what has gone before and to what is to follow is necessary for the proper appreciation of the different sections.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
November, 1905

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* In fact the preface to his most recent series of essays "La valeur de la science," Paris, 1905, contains the statement that the changes in science which have taken place in the last few years call for a revision of some of the ideas propounded in the book here under review.

† An English rendering, containing a preface by Larmor and published by the Walter Scott Publishing Company, is reviewed by Russell in *Mind*, July, 1905, pp. 412–417.

LA MÉCANIQUE STATISTIQUE.

Elementary Principles in Statistical Mechanics, Developed with especial Reference to the Rational Foundations of Thermodynamics. By J. WILLARD GIBBS. Yale Bicentennial Publications. New York, Scribner and Sons; London, Arnold; 1902. xviii + 207 pp.

PEUT-ÊTRE les lecteurs du BULLETIN ne s'étonneront-ils pas trop de l'époque tardive à laquelle ils liront le compte rendu des Elementary Principles. Non seulement — est-il besoin de le dire ?— ce livre n'est point de ceux que l'on analyse hâtivement; mais, d'autre part, les questions qu'il traite ont été vivement agitées dans ces derniers temps; les idées défendues par Gibbs ont fait l'objet de nombreuses controverses; les raisonnements par lesquels il les a appuyées ont, eux aussi, subi la critique. Il me paraît intéressant d'étudier son oeuvre à la lumière de ces controverses et en discutant ces critiques.

Gibbs craignait que sa Mécanique statistique ne semblât trop physique aux mathématiciens, trop mathématique aux physiciens. Nous ne ferons qu'énoncer un truisme en disant que ce scrupule exprime une des qualités précieuses de l'ouvrage. La nécessité d'associer, de la manière la plus intime possible, deux branches de la science qui s'étaient trop longtemps perdues de vue, n'est plus à démontrer, et il faut espérer qu'elle continuera à être comprise.

En fait, les physiciens ont déjà apprécié le livre de Gibbs et l'ont utilisé. On ne saurait mettre en doute son importance au point de vue des mathématiciens. Ceux-ci connaissent encore insuffisamment les théories cinétiques, où il semble que leur activité pourrait si utilement s'exercer. Non seulement ils ont besoin d'y être initiés; mais il ne leur est pas indifférent d'y être introduits par un des leurs, d'y trouver un guide qui d'abord s'appelle Gibbs et, ensuite, leur parle leur propre langue: on sait qu'à cette fusion des différentes branches de la science dont nous rappellions tout à l'heure l'importance, la discordance des points de vue auxquels elles se placent et la différence des éducations qu'elles supposent est un obstacle souvent plus sérieux que la multiplicité des connaissances qu'elles embrassent.

La "Mécanique statistique" est, en somme, une question purement mathématique. Elle n'est autre chose que l'application du calcul des probabilités à la mécanique, que la dynamique d'un grand nombre de systèmes régis (dans le cas le plus simple) par les mêmes équations, mais avec des circonstances initiales différentes. Ce qui est observable et, par conséquent, important, c'est la valeur moyenne de telle ou telle quantité, fonction de l'état du système. A cette recherche, on substitue, avec Maxwell, celle du nombre de systèmes pour lesquels la quantité ainsi envisagée est (à un instant donné quelconque) comprise entre des limites données. Ce nombre sera, en général, très grand,— si grand que par un raisonnement logiquement absurde, fécond et rarement trompeur en fait, (du moins lorsqu'on n'entend pas le poursuivre jusque dans ses dernières conséquences) on applique la loi des grands nombres, même lorsque les limites dont il s'agit sont infiniment rapprochées.

Que seront, individuellement, les systèmes S auxquels s'appliquerons ces considérations? Gibbs, à l'exemple de plusieurs de ses prédécesseurs, fait abstraction de toute supposition particulière à leur égard : il part des équations canoniques d'Hamilton sous leur forme la plus générale, avec les coordonnées de position $q_1, q_2, \cdots, q_n$ et les "moments généralisés" $p_1, p_2, \cdots, p_n$ substitués aux vitesses.

Il y a là autre chose qu'une généralisation toute théorique. D'une part on ne sait encore trop sous quelle forme on arrivera à se représenter ces éléments matériels ou électriques, dont les évolutions compliquées seront chargées de rendre compte des propriétés des corps naturels.

D'autre part, les principes de la mécanique, eux-mêmes, sont en train de se recodifier : voici que la masse n'est plus une constante, mais une fonction de la vitesse. Une seule chose semble devoir subsister. Les nouvelles équations pourront, comme les anciennes, être déduites du calcul des variations; la nouvelle mécanique admettra elle aussi, un *principe de moindre action*, que l'on s'est déjà occupé de formuler.*

Or cela suffit pour que les équations canoniques continuent à être valables.

Gibbs les considère d'une manière presque exclusive : il ne fait jouer qu'un rôle tout épisodique au cas particulier où il s'agit de points matériels isolés. Par là, son travail gagne singulièrement en importance. Il gagne aussi en simplicité: j'entends cette vraie simplicité qui consiste à débarrasser la question de tous les éléments inutiles. Je ne crois pas qu'une autre forme d'exposition soit aussi propre à montrer aux mathématiciens ce qu'ils ont surtout intérêt à savoir : la voie qu'ils doivent suivre pour le progrès de cette théorie qui est de leur domaine et qui s'est fondée sans leur concours. A la lecture des Elementary principles, cette voie se dégage clairement: c'est le développement des principes que la théorie générale des équations différentielles et la mécanique analytique doivent à M. Poincaré.

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* Voir entre autres, deux Notes récentes presentées à l'Académie des Sciences de Paris, l'une par MM. E. et Fr. Cosserat, l'autre par M. Langevin.

Tout d'abord, la notion *d'invariant intégral*, qui, dans les Méthodes nouvelles de la Mécanique céleste, a reçu les belles applications que l'on sait, a été mise dès l'abord à la base de la mécanique statistique. Les trois premiers chapitres de l'ouvrage qui nous occupe sont un commentaire de cette double proposition :

1° L'unité est un multiplicateur des équations canoniques :

2° On connaît donc un invariant intégral de ces équations.

Cet invariant intégral est le *volume* dans la terminologie de M. Poincaré. Gibbs lui donne le nom *d'extension en phase.* Une *phase* est un état du système S, caracterisé par sa position et ses vitesses, autrement dit, par les valeurs de $q_1, q_2, \cdots, q_n ; p_1, \cdots, p_n$ et que l'on peut, par conséquent, représenter par un point dans l'espace E_{2n} à $2n$ dimensions. Il arrivera d'ailleurs, en général, que ce point ne pourra pas occuper une position absolument quelconque dans l'espace E_{2n}, mais devra se trouver dans une certaine région limitée R_0 de cet espace. L'extension en phase est relative à une portion quelconque R de R_0: c'est l'intégrale

$$\int\int\cdots\int_R dq_1\cdots dq_n\, dp_1\cdots dp_n.$$

Si l'on donne la position du point représentatif au temps t, on détermine implicitement, par cela même sa position à un autre instant quelconque t'. Si on prend, pour le premier point, successivement chacun de ceux de la région R, le second point décrira une nouvelle région R'. Celle-ci a même extension en phase que la première : l'extension en phase est invariante dans le temps (sur une même série de trajectoires).

Mais Gibbs attire l'attention sur une autre propriété que possède ce même symbole et qui, moins remarquable sans doute que la première, n'en est pas moins des plus importantes. L'extension en phase est aussi invariante par rapport au système de coordonnées employé. Quelque soit le choix des q_i, du moment que les p_i sont choisis en conséquence, les extensions en phase seront conservées. On peut donc parler de *régions dynamiquement équivalentes** de l'espace à $2n$ dimensions — ce qui n'aurait pas de sens absolu sans la propriété d'invariance dont nous parlons (ce qui n'aurait aucun sens, par exemple, si les équations n'étaient pas canoniques).

On peut aller plus loin : Gibbs opère sur les positions seules, ou sur les vitesses seules, comme il a opéré sur l'ensemble des deux. Il peut ainsi définir une *extension en vitesse* et une *extension en configuration.*

Si étroitement liées à l'extension en phase que soient les deux notions précédentes, c'est intentionnellement que Gibbs en ajourne l'étude à son sixième chapitre. Il doit d'abord aller au plus pressé, au but principal du livre et introduire les notions statistiques sur lesquelles il raisonnera.

Grâce à la convention qui permet d'appliquer la loi de grands nombres à un élément infinitésimal, on représentera le nombre des systèmes S contenus (à un instant donné) dans une extension en phase donnée (c'est-à-dire dont le point représentatif est dans une région donnée R de l'espace E_{2n}) par l'intégrale

$$\int\int\cdots\int_R D\,dq_1\cdots dq_n dp_1\cdots dp_n,$$

où D est une fonction des q et des p, *la densité en phase.* La connaissance de la valeur de D en fonction de $q_1, \cdots, q_n ; p_1, \cdots, p_n$ donne la distribution absolue des systèmes donnés entre leurs différentes phases. En général, la distribution relative, proportionnelle, intervient seule, et il suffit d'étudier, N étant le nombre total des systèmes S, le quotient

$$P = D/N$$

ou *coefficient de probabilité, la probabilité en phase* étant l'intégrale

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* L'extension en phase est une notion de dynamique (et non de cinématique) quoique les forces n'y entrent pas, parce qu'elle dépend des masses.

$$\int\int\cdots\int Pdq_1\cdots dq_n\,dp_1\cdots dp_n.$$

Enfin la quantité

$$\eta = \log P$$

est *l'indice de probabilité.*

L'équilibre moyen d'un ensemble de systèmes S tel que celui que nous considérons — ou, plus généralement, le régime permanent — correspond à un *équilibre statistique,* caractérisé par ce fait que la distribution en phase est indépendante du temps.

Toutes les quantités que nous avons définies plus haut, probabilité en phase, coefficient de probabilité, indice de probabilité, qui sont invariantes dans le temps sur chaque trajectoire, sont des intégrales des équations différentielles. La condition d'équilibre statistique est que ces intégrales ne contiennent pas t.

Gibbs considère, en particulier, la distribution dans laquelle l'indice de probabilité est une fonction, et même une fonction linéaire, de l'intégrale la plus simple, l'énergie : c'est la *distribution canonique,* qu'il introduit ainsi à priori (et qui d'après ce qui précède, reste telle dans tout système de coordonnées).

La probabilité en phase est-elle, en réalité, une fonction de l'énergie seule? C'est, on le sait, une des questions qui ont été le plus fréquemment discutées dans cette théorie. Gibbs montre comment elle se pose. Un exemple simple fait voir que, au moins à priori, des intégrales autres que l'énergie peuvent jouer le même rôle et régler les valeurs de la probabilité en phase. Il suffit de considérer des points matériels placés, soit dans une sphère donnée, soit (c'est le cas considéré par Gibbs) entre deux sphères concentriques, et rendus incapables de traverser la ou les sphères limites. Ce dernier résultat s'obtient pour Gibbs, en faisant intervenir des forces centrales attractives ou répulsives suffisamment intenses ; on peut aussi imaginer que ces sphères soient des parois solides le long desquelles les points considérés se réfléchissent (élastiquement). L'hypothèse de Gibbs a l'avantage de faire intervenir des vibrations qui offrent une certaine analogie avec la théorie des radiations. Mais, dans l'un ou l'autre cas, les moments cinétiques par rapport à trois axes rectangulaires issus du centre sont constants pour chaque point : rien n'empêchera de concevoir la probabilité en phase comme dépendant, non seulement de l'énergie, mais encore de ces trois moments ou, du moins — si l'on veut sauvegarder l'isotropie de l'ensemble — de la somme de leurs carrés.

Rien ne dit même que les choses ne se passent pas de façon plus compliquée encore. Tout ce que l'on peut dire en considérant, à priori, la probabilité en phase, comme une fonction régulière est qu'elle constitue une intégrale *uniforme* des équations de la dynamique. Mais *l'uniformité* n'a pas ici le même sens que dans les recherches de M. Poincaré, puis qu'il ne s'agit plus nécessairement de fonctions analytiques. Elle serait encore, cependant, de nature à restreindre d'une manière notable la catégorie des expressions acceptables pour P.

Quant au mode de continuité de cette fonction, nous ne possédons qu'une proposition capable — autant qu'on en peut juger actuellement — de nous éclairer à ce sujet : c'est celle que M. Poincaré a déduite de la notion d'invariant intégral. D'après ce théorème, que Gibbs énonce à une autre occasion (au Ch. XII), les solutions des équations de la dynamique possèdent, en général, *la stabilité à la Poisson*, c'est-à-dire que le point représentatif M repasse, en général, une infinité de fois dans le voisinage (aussi immédiat qu'on le veut) de toute position M_0 déjà occupée. Les trajectoires pour lesquelles cela n'a pas lieu sont *exceptionnelles :* leurs points de départ ne peuvent former une extension en phase finie, si petite soit-elle.

Si le point M pouvait s'approcher de M_0 dans toute direction tangente à la multiplicité $\epsilon =$ const. (ϵ désignant l'énergie), il n'y aurait point d'autre intégrale uniforme possédant des dérivées partielles que l'intégrale des forces vives. Si M pouvait, dans le cours du mouvement, passer infiniment près de tout point de cette même multiplicité, l'intégrale des forces vives serait la seule intégrale continue.

Ce n'est pas, bien entendu, ce qui à lieu dans l'exemple choisi tout-à-l'heure. Mais n'en est-il pas autrement dans des problèmes moins particuliers? Cette question, de pur calcul intégral, reste sans solution mathématique. Les physiciens sont obligés de la discuter comme bien d'autres — voire, de l'aborder expérimentalement, ainsi que l'a fait Lord Kelvin — sans attendre notre jugement sûr, mais par trop tardif.

Après avoir indiqué que la distribution canonique peut ne pas être la seule imaginable, Gibbs retourne à cette distribution canonique, pour y mettre en évidence les propriétés que possède le coefficient de ϵ dans l'expression linéaire

$$\eta = \frac{\psi - \epsilon}{\Theta}$$

(qui représente l'indice de probabilité η) ou plutôt l'inverse Θ de ce coefficient. *Ces propriétés sont tout analogues à celles de la température absolue.* Par exemple la quantité Θ se comporte comme la température dans le mélange de deux ensembles ; et, surtout, elle donne lieu, lorsqu'on la multiplie par $d\epsilon$ et qu'on tient compte, s'il y a lieu, des forces extérieures, à une équation toute semblable à celle que donne le principe de Carnot. Le rôle de l'entropie (changée de signe) est alors joué par la valeur moyenne $\bar{\eta}$ de l'indice de probabilité.

C'est seulement à ce moment que sont introduites les notions, mentionnées tout à l'heure, d'extension en configuration et d'extension en vitesse et comme conséquence, l'étude des valeurs moyennes (ainsi que des anomalies correspondantes) qui se rapportent non plus seulement à l'énergie totale, mais à l'énergie cinétique et à l'énergie potentielle, considérées isolément.

Les quantités Θ, $-\bar{\eta}$, que nous avons vues jouer respectivement, dans une distribution canonique, le rôle de la température et celui de l'entropie, sont-elles seules susceptibles de correspondre à ces deux notions? Et si la distribution cesse d'être canonique, sommes-nous sûrs que l'entropie sera encore (aux signes près) la valeur moyenne de l'indice de probabilité?

La réponse, au moins sous réserve de considérations nouvelles, est négative. On peut former d'autres expressions de formes différentes, coïncidant d'ailleurs avec les premières dans le cas de la distribution canonique et possédant, elles aussi, les deux propriétés auxquelles nous avons trouvé plus haut une analogie thermodynamique.

Pour arriver à ce résultat, Gibbs considère toutes les phases pour lesquelles l'énergie est inférieure à une certaine quantité ϵ. L'extension en phase V de l'ensemble des états ainsi définis est une certaine fonction de ϵ, dont la dérivée est désignée par e^{ϕ}.

Il est clair que les fonctions V et ϕ de l'énergie ϵ ont une importance toute particulière. Leur introduction dans l'étude d'un ensemble canonique revient à classer les systèmes S dont il se compose d'après leurs énergies, à le décomposer, autrement dit, en parties dont chacune contient les systèmes pour lesquels cette énergie est comprise entre des limites données. Si ces limites sont suffisamment rapprochées, on peut regarder, dans l'intervalle qu'elles déterminent, l'énergie comme constante et on a la distribution que Gibbs appelle *microcanonique,* sorte d'élément de la distribution canonique.

Ce sont des combinaisons diverses des fonctions V et ϕ qui dans le cas de la distribution microcanonique, puis même (par la considération de leurs valeurs moyennes) en dehors de ce cas, fournissent des quantités substituables à Θ et à $-\bar{\eta}$.

Il reste à traiter la question la plus importante et la plus délicate que soulève cette étude de la distribution en phase. Que devient cette distribution au cours du mouvement, lorsqu'on part d'un état quelconque: tend-elle, par exemple, à se rapprocher de la distribution canonique ou d'une distribution présentant des propriétés analogues?

C'est, en somme, la question vitale pour les théories cinétiques. Le paradoxe qui s'y rattache et qui semble, au premier abord, miner par avance toute théorie de cette nature est en effet le suivant. Comment, en partant d'équations de la dynamique, toutes *réversibles*, parviendra-t-on à des lois *irréversibles*, à la croissance de l'entropie?

Gibbs se demande donc—après avoir établi une série de propriétés de minimum qui doivent jouer un rôle essentiel dans cette étude — ce que devient le mouvement lorsqu'on le considère à de longs intervalles de temps. Il commence par indiquer, à cette occasion, le théorème de M. Poincaré sur la stabilité à là Poisson auquel nous faisions allusion un peu plus haut.

La contradiction apparente qu'il s'agit d'élucider est ensuite posée, sous une forme nouvelle et plus frappante encore que celle qui a été rappelée tout à l'heure. Cette entropie qui doit être croissante, les calculs précédents conduisent, comme le veut d'ailleurs la théorie de M. Boltzmann, à la rattacher à la valeur moyenne de l'indice de probabilité. Or nous savons que cet indice est constant dans le temps. Comment sa valeur moyenne pourra-t-elle varier de quelque manière que ce soit?

La comparaison employée par Gibbs montre avec évidence comment des singularités de cette nature s'introduisent nécessairement dans l'application des considérations statistiques à des systèmes dont les différentes parties se mélangent entre elles.

Soit un vase de forme cylindrique (dont, pour simplifier, on pourra considérer exclusivement une section droite) et supposons ce vase rempli de deux liquides A et B de telle manière qu'à l'origine des temps ces liquides soient distribués par secteurs: par exemple, deux diamètres perpendiculaires seront supposés diviser le cercle en quatre parties dont deux opposées contiendront exclusivement du liquide A et les deux autres du liquide B.

Imaginons maintenant que les différentes couches circulaires concentriques dont est composé le milieu soient animées chacune d'un mouvement de rotation, mais avec des vitesses angulaires différentes. Si nous suivons dans son mouvement une portion quelconque du liquide A, par exemple, son volume ne changera pas et sa densité restera également invariable. La connexion de ce volume ne sera même modifiée à aucun moment.

Cependant, une portion déterminée quelconque de l'espace intérieur au vase, si petite qu'elle soit et qui, à l'origine, renfermait exclusivement du liquide A ou du liquide B, renfermera pour peu qu'on suive le mouvement pendant un temps suffisamment prolongé un mélange des deux fluides, mélange sensiblement homogène* pour t suffisamment grand. En particulier la densité, tout invariante qu'elle soit, ne sera plus nulle part celle de A ou celle de B, mais aura pris une valeur intermédiaire.

Si l'on suit le mouvement à partir de l'origine, on verra cette densité s'égaliser ; l'homogénéité du liquide (et il est clair qu'on pourrait, de bien des manières, définir un nombre qui serve de mesure à cette homogénéité) ira d'une manière générale croissant.

Le parallélisme est, on le voit, complet avec le phénomène qui est en question pour la quantité $\bar{\eta}$, et, plus loin, Gibbs sera conduit à préciser et à montrer, comme Boltzmann, que cette quantité doit être décroissante.

Seulement, comme l'a remarqué M. Brillouin, il ne faut pas se hâter de se féliciter de la concordance qui existe entre les résultats des deux auteurs. Ils ont, en réalité, des sens différents, et on pourrait même être tenté de les regarder comme presque contradictoires.

L'analyse de M. Boltzmann † repose essentiellement sur l'intervention des *chocs :* la condition nécessaire et suffisante pour qu'elle soit valable est que ces chocs se produisent en grand nombre : il en résulte même qu'elle peut être applicable dans un intervalle de temps très court, si le nombre des molécules est assez considérable pour cela.

Gibbs ne considère pas les chocs : nous sommes arrivé au XII^e Chapitre de son ouvrage sans qu'il y ait fait la moindre allusion. Ses démonstrations concernent des ensembles composés de systèmes sans action les uns sur les autres, ensembles *différents*, par conséquent, de ceux de Boltzmann. Elles font intervenir, au lieu des chocs, des phénomènes de mélange analogues à ceux auxquels nous avons fait allusion tout-à-l'heure pour le cas des liquides. Quelque soit le nombre des systèmes, elles ne s'appliquent qu'à un intervalle de temps $t' - t$ relativement long, assez long pour que deux systèmes S très voisins au temps t puissent correspondre, au temps t', à des points représentatifs très différents, ou inversement.

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* Ou du moins, dont la densité n'est fonction que de la distance au centre.

† Voir Leçons sur la théorie des gaz, tome I.

Malgré cela, nous verrons plus loin que l'identité des deux conclusions n'est probablement pas fortuite, qu'elle est, au contraire, dans la nature des choses. Mais ceci nous amène à un sujet important, aux objections qu'à l'exemple de M. Brillouin * beaucoup de physiciens ont opposées à ces idées de Gibbs et de Boltzmann.

Il est reproché tout d'abord à la démonstration de M. Boltzmann de ne considérer l'effet des chocs que sur les vitesses et de supposer constamment l'homogénéité parfaite en ce qui regarde les positions, alors que cette homogénéité — à supposer qu'elle existât à un instant quelconque — serait manifestement détruite aux instants suivants par le mouvement lui-même. On introduirait ainsi constamment l'inorganisation parfaite (à savoir, l'homogénéité) à un point de vue et c'est ainsi qu'au bout du compte cette inorganisation paraîtrait être la tendance générale du mouvement, tendance qui n'existerait nullement en réalité. En un mot, l'analogie de Boltzmann, radicalement fausse, ne serait pas une réponse à la contradiction signalée plus haut.

M. Brillouin écarte à plus forte raison le raisonnement de Gibbs, image plus grossièrement approchée de la réalité que celui de Boltzmann et qui même, nous l'avons vu, porte sur des phénomènes essentiellement différents.

Je serais, pour ma part, tenté de donner raison à Gibbs et à Boltzmann contre mon savant ami M. Brillouin. Je vais essayer ici d'expliquer pourquoi et de faire à sa critique pénétrante une réponse que le lecteur ne s'étonnera sans doute pas de trouver longue.

Remarquons d'abord que la question est double. La conclusion contestée se compose de deux parties, une négative et une positive. Elle exprime :

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* Notes pour la traduction française des Leçons sur la théorie des gaz de Boltzmann, Paris, Gauthier-Villars, 1901–05.

1° que l'objection tirée de la réversibilité des équations du problème n'est pas valable, qu'elle ne prouve pas l'impossibilité de lois finales irréversibles ;

2° qu'en fait, de telles lois irréversibles existent et qu'une certaine quantité (dont on donne l'expression) va en général en croissant.

Or, tant qu'on se borne à affirmer la première partie c'est-à-dire à prétendre que la seconde est *possible*, il ne paraît guère y avoir de doute. Dans ce cas, en effet, il importe peu que les raisonnements présentés correspondent à la réalité physique ; il suffit qu'ils portent sur des phénomènes logiquement concevables. Dès lors, les déductions de Gibbs et de Boltzmann ne sont pas, à ce point de vue, atteintes par les objections précédentes. On peut, au reste, se dispenser d'examiner longuement la question en recourant aux exemples les plus simples, tels que celui des projectiles, la pesanteur étant considérée comme un champ de forces constant en grandeur et en direction. Là encore, les équations du problème sont réversibles ; et cependant si un certain nombre de tels projectiles sont lancés *au hasard*, qui doutera que d'une manière globale, ils n'aillent en descendant ?

Soit encore un système sans forces accélératrices dépendant de deux paramètres ρ, θ, et dont la force vive est donnée, par

$$2T = f(\rho)\left(\frac{d\rho}{dt}\right)^2 + \phi(\rho)\left(\frac{d\theta}{dt}\right)^2.$$

Supposons que $f(\rho)$, mais non $\phi(\rho)$, soit très grand pour ρ voisin d'une certaine valeur ρ_0 et très petit partout ailleurs. Alors toute trajectoire de ce système restera, sauf pendant des intervalles de temps très petits et très éloignés, au voisinage de la ligne $\rho = \rho_0$. Dès lors la différence $|\rho - \rho_0|$ sera en général décroissante pour tout intervalle de temps non très petit, si elle n'est pas initialement très petite. Il est clair qu'un tel résultat n'est irréversible qu'en apparence ; il tient évidemment à ce que les régions du plan (ρ, θ) non très voisines de $\rho = \rho_0$ doivent être considérées comme très petites au point de vue du problème actuel : il y aurait, autrement dit, lieu de définir ici, d'une manière plus ou moins analogue à celle qui a été mentionnée plus haut, des *régions dynamiquement équivalentes* et cette définition serait très différente de la définition géométrique. Nous trouverons d'ailleurs plus loin le point précis par lequel pèche, dans des exemples de cette espèce, l'objection générale.

Employons auparavant une autre comparaison plus lointaine en apparence, beaucoup plus exacte cependant à mon avis. Considérons ce qui se passe lorsqu'on bat un jeu de cartes. C'est ce que l'on fait le plus généralement par une série de coupes successives, l'opération élémentaire consistant à diviser le jeu, d'une manière arbitraire, en trois parties et à intervertir l'ordre de deux consécutives d'entre elles. Il est clair qu'une telle manière de procéder est parfaitement réversible. Cependant il est non moins évident qu'elle tend à une certaine régularisation du jeu. Si, par exemple, une fois les cartes battues nous trouvions toutes les rouges d'abord et toutes les noires ensuite, nous croirions à une fraude. On peut mesurer cette régularisation par un nombre. Considérons, pour fixer les idées, le nombre V de cartes rouges suivies d'une noire. Il est aisé de calculer la valeur moyenne V_0 de V pour tous les ordres possibles dans lesquels les cartes peuvent être rangées. Si maintenant, on part d'un jeu donné pour lequel V a une valeur déterminée V', la valeur probable de ce nombre après battage se rapprochera de V_0. J'entends par là que si au lieu d'un seul jeu nous en considérons un très grand nombre, tous identiques entre eux initialement, mais battus de toutes façons possibles et au hasard, la moyenne des valeurs de V pour ces différents jeux sera comprise entre V' et V_0.

Cette opération du battage est tout à fait comparable aux phénomènes qui ont lieu pour nos systèmes S, même lorsqu'on reste au point de vue de Gibbs. La position P' d'un point représentatif au temps t' étant une fonction *pratiquement discontinue* de la position P du même point au temps t, lorsque l'intervalle $t' - t$ est très long (c'est-à-dire qu'un changement très petit de P produit un changement très grand de P'), tout se passe comme si, P étant donné, P' était plus ou moins indéterminé, comme si la donnée de P équivalait au lancé d'une roulette dont l'arrêt marquerait la position de P'. Cette roulette fonctionnerait aussi parfaitement que possible si l'indétermination de P' était absolue (sauf la condition $\varepsilon =$ const.)

C'est l'hypothèse dont nous avons parlé précédemment et sur le bien fondé de laquelle on ne peut se prononcer avec certitude. Mais même si elle était inexacte, il n'en est pas moins vrai que des points représentatifs primitivement rapprochés les uns des autres se dispersent avec le temps et que notre ensemble de systèmes S est ainsi " battu."

Il est clair également que les chocs agissent dans le même sens et d'une manière beaucoup plus efficace encore. Nous voyons maintenant comment les procédés si différents de Gibbs et de Boltzmann peuvent conduire au même résultat. C'est ainsi que la valeur moyenne du nombre V considéré plus haut tendrait encore vers V_0 si l'on employait pour battre les cartes, au lieu du procédé auquel nous avons fait allusion, un autre quelconque de ceux qui sont usités dans ce but.

Ceci tient, remarquons-le, à ce que ces procédés si différents qu'ils puissent être, ne favorisent pas une permutation plutôt qu'une autre. Il faudrait donc savoir de même dans le cas de nos systèmes S, quelles sont les distributions également favorisées et leur définition dépendra du mode de "battage." Toutefois l'exemple même qui précède montre que cette définition pourra être la même pour des modes de battage très différents.

Passons à la seconde partie de la thèse, à la question de savoir si, en effet, la valeur moyenne de l'indice de probabilité changé de signe est en général croissante.

Du moment que M. Boltzmann est obligé d'attribuer aux molécules d'autres positions que celles qu'elles ont en réalité, son raisonnement doit, si l'on veut prendre les choses à la lettre, être considéré comme inexistant.

Disons tout de suite que Gibbs, de son côté, n'apporte pas de démonstration rigoureuse.

Non seulement il admet l'absence de chocs, mais — si j'ai bien compris son raisonnement — il ne tient pas suffisamment compte du phénomène même de mélange sur lequel il se base. Ce phénomène a, en effet, pour résultat, me semble-t-il, de rendre l'application de la loi des grands nombres aux extensions en phase infiniment petites (fondement premier de toute cette analyse) sinon douteuse du moins plus délicate que la brève rédaction de Gibbs ne le comporterait. La conclusion même serait de nature à faire mettre en doute la légitimité de la démonstration, en vertu du principe "qui veut trop prouver ne prouve rien." Elle n'admet, en effet, d'autres cas d'exception que ceux où il n'y a pas mélange, où les équations différentielles données définissent des trajectoires toutes stables au sens de M. Liapounoff: l'inégalité qui exprime le théorème serait alors remplacé par une égalité. Or il y a lieu, en l'espèce, de prévoir des cas d'exception beaucoup plus étendus, correspondant non à des formes particulières des équations différentielles, mais à certaines distributions initiales des systèmes S, et pour lesquels l'inégalité serait *renversée.*

Mais si nous ne nous autorisons pas des raisonnements de Gibbs et de Boltzmann pour énoncer une affirmation rigoureuse, quelles présomptions ces raisonnements créent-ils ? S'ils sont défectueux dans quel sens pèchent-ils ?

Devons-nous, par exemple, croire avec M. Brillouin qu'en introduisant l'homogénéité, on introduit du même coup l'inorganisation ? Je ne le pense pas. Le sens du mot "inorganisé" demanderait à être défini ; mais, en tout cas, la distribution la plus "inorganisée" possible n'est pas plus la distribution homogène que n points pris au hasard sur une circonférence n'ont de chance pour être les sommets d'un polygone régulier. Pour revenir à notre comparaison de tout à l'heure, un jeu de cartes, une fois battu, où les rouges seraient toutes réunies ainsi que les noires, serait suspect ; mais un jeu où il y aurait alternance parfaitement régulière entre les uns et les autres ne le serait pas moins. Si — ce qui est certain — une distribution homogène des systèmes S dans l'espace à n dimensions à un instant t est remplacée aux instants suivants par une distribution non homogène, celle-ci est, selon toute vraisemblance, beaucoup plus inorganisée que la première. Non seulement, si erreur il y a, c'est dans le sens de l'organisation, mais on pouvait être presque sûr à priori, qu'il en serait ainsi. Une erreur de cette espèce a, en effet, toujours pour origine le désir de simplifier le raisonnement. Or simplifier, en mathématique, c'est débrouiller un peu le chaos des hypothèses possibles : en un mot, c'est organiser.

Quant à une définition satisfaisante de l'inorganisation, elle est évidemment difficile à donner. Je proposerai, faute de mieux, l'une ou l'autre des deux suivantes.

On peut d'abord * considérer comme type de distribution inorganisée l'ensemble de toutes les distributions possibles des systèmes S. Une distribution déterminée D serait alors dite inorganisée si elle ne se distinguait en rien, *au point de vue de la question étudiée*, de la distribution totale dont nous venons de parler. On pourrait donc dans le raisonnement substituer cette dernière à la distribution considérée, — par exemple, prendre pour valeur d'une quantité qui dépend de la distribution D, sa valeur moyenne dans toutes les distributions possibles — et c'est la possibilité de cette substitution, qui exprimerait le caractère inorganisé de D.

Dans cette conception, une distribution pourrait — conformément à la nature des choses—être considérée comme inorganisée relativement à telle question et comme organisée relativement à telle autre. Cela peut être le cas, par exemple, pour la distribution homogène (en position) des systèmes S: il est possible que l'organisation ainsi introduite soit sans influence dans la question qui nous occupe.

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* C'est ce que propose Gibbs à la fin du chap. XI (p. 251).

On peut aussi admettre simplement que les distributions "organisées" sont des distributions *exceptionnelles*, au sens adopté par M. Poincaré à propos de son théorème sur la stabilité à la Poisson. L'hypothèse, faite au cours du raisonnement, que le mouvement est inorganisé, reviendrait alors à dire, sans plus préciser, que le théorème est vrai sauf pour certaines distributions exceptionnelles.

L'une ou l'autre des deux définitions précédentes suppose d'ailleurs la connaissance de ce que l'on doit appeler distributions "également probables." Toutefois, on sait que la notion de distributions exceptionnelles est en somme indépendante de la convention adoptée à cet égard.

Mais la notion de mouvement inorganisé absolument parlant, n'est pas la seule qu'il convienne d'introduire. Il y a lieu d'y joindre, suivant un procédé classique en calcul des probabilités, celle de mouvement inorganisé parmi ceux qui satisfont à des conditions données.

L'adaptation, à cet effet, des deux définitions données tout à l'heure ne souffre évidemment aucune difficulté. Si j'insiste sur cette extension c'est qu'elle permet à mon sens de mettre en évidence le véritable vice de l'objection que je m'étais proposé de discuter.

Celle-ci repose, en effet, sur ce fait, que tout mouvement qui a existé du temps t au temps postérieur t' peut être renversé, le temps t' étant considéré comme antérieur à t et que, *inorganisé dans le premier cas, il le sera également* dans le second.

C'est ce dernier point qui ne me paraît pas exact.

Reprenons le cas des projectiles pesants, en supposant donnée, pour fixer les idées, la constante des forces vives. Considérons une série de ces projectiles lancés au hasard à une même hauteur h et dont, par conséquent, les mouvements ne seront pas exceptionnels, au moins à ce moment-là. Tous passeront ensuite à la hauteur h', celle-ci étant supposée inférieure, et inférieure d'une quantité très grande, à h. Mais, dans cette nouvelle situation de nos projectiles, leurs mouvements seront devenus exceptionnels : ils seront tous sensiblement verticaux et descendants.

L'explication du paradoxe est immédiate : au premier instant, nos mobiles ne sont pas exceptionnels parmi ceux qui sont à cet instant-là à la hauteur $z = h$; au second instant, ils sont exceptionnels parmi ceux qui (toujours avec la vitesse correspondante à la valeur donnée de la constante des forces vives) sont à la hauteur h' : question toute différente de la première.

Autrement dit, le fait, pour un mouvement, d'être organisé ou inorganisé dépend de ce que l'on suppose préalablement connu à son sujet.

Si donc H est une certaine fonction de la distribution des systèmes S, la propriété suivante :

"La quantité H est croissante si la distribution est inorganisée"

est susceptible d'un énoncé du genre de celui-ci, où je ne vois plus de prise pour l'objection de réversibilité :

> "Soient H_1 et $H_2 < H_1$ deux valeurs de H; T un intervalle de temps convenablement choisi.
>
> "Désignons par M_1 les mouvements pour lesquels, pendant l'intervalle T, la quantité H prend au moins une fois la valeur H_1 ; par M_2, les mouvements pour lesquels (au cours du même intervalle de temps) H prend au moins une fois la valeur H_2 ; par M_3, ceux qui satisfont à l'une et à l'autre des deux conditions précédentes ; autrement dit, qui sont à la fois des mouvements M_1 et des mouvements M_2.
>
> "Les M_3 sont exceptionnels parmi les M_1, mais non parmi les M_2."

Je ne doute guère, pour ma part, de l'existence d'un pareil énoncé après les déductions de Gibbs et de Boltzmann ; et si leur conclusion était erronée, l'erreur ne me paraîtrait pouvoir porter que sur l'expression de la quantité H.

Je me suis laissé entraîner à commenter longuement cette partie si essentielle de l'oeuvre de Gibbs, et je craindrais d'abuser de l'attention du lecteur en insistant sur les trois derniers chapitres.

Nous avons, au reste, déjà parlé du sujet — le plus important peut-être après celui, qui vient de nous occuper — auquel se rapportent les chapitres XIII et XIV : les définitions de la température et de l'entropie, dans l'hypothèse cinétique : définitions qui sont examinées — toujours, il est vrai, en admettant la distribution canonique ou microcanonique, à la lumière des principales propriétés des deux notions à définir.

Le dernier chapitre de l'ouvrage mériterait une attention spéciale. Gibbs y considère des ensembles ("grands ensembles") dont les systèmes constituants sont eux-mêmes des ensembles ("petits ensembles") de molécules. Le nombre de ces molécules varie d'ailleurs d'un "petit ensemble" à un autre, comme leur positions et leurs vitesses; et, d'autre part, on admet l'échange des particules entre les différents "petits ensembles." C'est dans les phénomènes de diffusion, particulièrement dans ceux qui ont lieu au contact des membranes semi-perméables, que Gibbs fait intervenir cette nouvelle catégorie d'ensembles, laquelle présente, on le voit, un caractère nouveau et important au point de vue théorique, puisque les différents systèmes partiels cessent d'être entièrement indépendants.

Telle est, dans ses grandes lignes, la dernière oeuvre de Gibbs. Il ne pourrait en exister de plus actuelle et de plus importante, en dépit de la modestie de son titre. Cette modestie n'est d'ailleurs pas de pure forme: elle correspond bien à la pensée arrêtée de l'auteur. Gibbs, nous l'avons vu, n'a pas abordé la théorie des chocs: il a voulu, sauf en son dernier chapitre, se borner au cas le plus simple, celui où le mouvement d'un des systèmes n'est pas troublé par la présence des autres. Son but n'a pas été, en un mot, de remplacer des ouvrages comme les Leçons sur la théorie des gaz de Boltzmann, mais de leur servir de base, ainsi qu'aux autres études qu'on pourra entreprendre sur le même sujet. C'est en ce sens surtout, peut-être, qu'il a fait oeuvre de mathématicien; cette sorte de besogne préparatoire est, il faut nous l'avouer, la seule qui nous incombe le plus souvent. Encore si nous pouvions la mener assez vite pour fournir aux besoins des sciences de la nature; mais rien que dans les Principes élémentaires de mécanique statistique, combien de questions avons-nous rencontrées auxquelles notre théorie des équations différentielles est actuellement incapable de répondre! Cependant pour effacé qu'il soit, ce rôle n'en est pas moins essentiel. Les successeurs de Boltzmann ne pourront se dispenser d'utiliser l'ouvrage de Gibbs: lui seul, en maintes circonstances, peut leur assurer l'instrument commode — et, avant tout, l'instrument sûr, dont ils ont besoin.

JACQUES HADAMARD.

NOTES.

AT the annual meeting of the London mathematical society held on November 9, the following officers were elected: A. R. FORSYTH, president; A. E. H. LOVE and J. H. GRACE, secretaries; J. LARMOR, treasurer; two vice-presidents and ten other members of the council. The following papers were read: By Mr. G. H. HARDY, "The continuum and the second number class"; by Professor W. BURNSIDE, "On the arithmetic nature of the coefficients in a group of linear substitutions of finite order (second paper)"; by Mr. J. W. NICHOLSON, "On the asymptotic value of a type of finite series"; by Professor T. J. I'A. BROMWICH, "On the extension of Dirichlet's integral"; by Dr. E. W. HOBSON, "On improper multiple integrals," and "On the arithmetic continuum."

AT the Meran meeting of the German mathematical society Professor A. PRINGSHEIM was elected president, and Professor A. KRAZER was re-elected secretary. Professors E. STUDY, of Bonn, and A. v. BRILL, of Tübingen were elected to the council. The society now numbers 666 members.

THE annual meeting of the association of teachers of mathematics of the Middle States and Maryland was held on Saturday, December 2, at Annapolis, Md., in affiliation with the association of colleges and preparatory schools of the Middle States and Maryland. The following papers were read: "How should the college teach analytic geometry?" by Professor H. S. WHITE; "Suggestions for the first twelve lessons in demonstrative geometry," by Mr. H. R. HIGLY; "Some essentials of the successful mathematics teacher," by Dr. J. S. FRENCH; "The teaching of geometry," by Dr. H. A. CONVERSE.

The officers elected for the coming year were: President, Professor E. S. CRAWLEY, University of Pennsylvania; Vice-president, Dr. J. S. FRENCH, Jacob Tome Institute, Port Deposit, Md.; Secretary and Treasurer, Dr. J. T. RORER, Central High School, Philadelphia, Pa. The association adopted the following resolution:

That this association approve of the organization of a national federation of existing associations of teachers of mathematics in which each association shall preserve its own organization and individuality and which shall have among its objects the joint support of publication. In the federation should be included only societies representing territory as extensive at least as one state.

THE fifth annual meeting of the central association of science and mathematics teachers was held at Chicago December 1 and 2. The following papers in mathematics were read: "The straight line in geometry," by J. W. WITHERS; "Interest and progress in the teaching of mathematics," by N. J. LENNES; "Aims in teaching algebra," by R. J. ALEY; "Some thoughts on the teaching of geometry," by C. A. PETTERSON.

THE first award of the new Bolyai prize recently founded by the Hungarian academy of sciences (see current volume of BULLETIN, page 43) was made to Professor H. POINCARÉ. The works of Professor D. HILBERT not only received honorable mention, but the report of the commission will contain a detailed account of his memoirs, as well as of those of the winner of the prize.

THE Munich technical school announces the following prize problem: "To develop historically the theory of the singular solutions of total differential equations of the first order from the first appearance of the theory to the present time. For the older theory, the memoir of Houtain, 'Des solutions singulières des équations différentielles' (*Annales des Universités de Belgique*, années 1851–1854) is to be critically discussed." Competing memoirs should be sent to the Senat der Technischen Hochschule zu München before October 15, 1906.

THE proceedings of the international congress of arts and sciences held at St. Louis in September, 1904, are being published by Houghton, Mifflin and Company in eight volumes, the first of which will contain the papers in philosophy and mathematics. The papers will be published as presented, except that those read in foreign languages will be translated into English.

WITH the appearance of its fifteenth volume, the publication of the *Jornal de Sciencias mathematicas e astronomicas*, edited by Professor F. G. TEIXEIRA, has been discontinued. In its place appears, under the direction of Professor Teixeira, the *Annaes Scientificos da Academia Polytechnica do Porto*, of which the first number has just been issued.

THE firm of Macmillan and Bowes, 1 Trinity St., Cambridge, England, announces that a series of short tracts of from 30 to 70 pages on various topics of pure mathematics and theoretical physics is soon to appear, edited by Mr. J. G. LEATHAM and Mr. E. T. WHITTAKER. The chief purpose is to assist in the maintenance of a high standard in mathematical teaching by the infusion of new methods and by the extension of knowledge of recent research. The following will shortly be published: Surface and volume integrals in physics, by J. G. LEATHAM; The integration of functions of a single variable, by G. H. HARDY; Quadratic forms and their classification by means of invariant factors, by J. I'A. BROMWICH; Definite integral, meaning and properties, by E. W. HOBSON; Singular points and asymptotes of plane curves, by C. A. SCOTT; The axioms of geometry, by A. N. WHITEHEAD; The eikonal and its application to optical instruments, by E. T. WHITTAKER.

THE publishing house of G. J. Göschen in Leipzig announces the following extensions of the Sammlung Schubert: "Theorie der ebenen algebraischen Kurven höherer Ordnung," by Dr. H. WIELEITNER, which appeared in November, and "Gewöhnliche Differentialgleichungen beliebiger Ordnung," by Professor J. HORN, which will be ready soon. A second volume on plane curves, treating special and transcendental curves has been promised by Dr. Wieleitner.

THE following university courses in mathematics are announced:

CAMBRIDGE UNIVERSITY. — *Michaelmas term*, 1905. — By Professor A. R. FORSYTH: Partial differential equations, three hours. — By Professor Sir G. H. DARWIN: Theory of potential and attractions, three hours. — By Professor Sir R. S. BALL: Planetary theory, three hours. — By Professor J. LARMOR: Electricity and magnetism, three hours. — By Professor J. J. THOMSON: Properties of matter, three hours; Electricity and matter, two hours. — By Professor B. HOPKINSON: Applied mathematics, two hours; Electricity, two hours. — By Dr. E. W. HOBSON: Vibrations and sound, three hours. — By Dr. H. F. BAKER: Introduction to theory of functions, three hours; Solid geometry, three hours. — By Mr. H. W. RICHMOND: Analytic geometry, three hours. — By Mr. E. T. WHITTAKER: Theory of optical instruments, three hours. — By Dr. A. N. WHITEHEAD: Principles of mathematics. — By

Mr. A. Berry: Elliptic functions, Bessel functions and Fourier series, three hours. — By Mr. A. Monro: Hydrodynamics and sound, three hours. — By Mr. J. G. Leathem: Elementary electron theory, three hours. — By Mr. J. H. Grace: Invariants and geometric applications, three hours. — By Mr. E. W. Barnes: Gamma functions, three hours.

Lent term, 1906. — By Professor A. R. Forsyth: Partial differential equations, II., three hours. — By Professor Sir G. H. Darwin: Dynamical astronomy (elementary), three hours. — By Professor Sir R. S. Ball: An elementary course on quaternions, three hours. — By Professor J. Larmor: Electrodynamics with optical applications, three hours. — By Professor J. J. Thomson: Electricity and magnetism, three hours; Discharge of electricity through gases, two hours. — By Professor B. Hopkinson: Applied mathematics II, two hours; Electricity II, two hours. — By Dr. E. W. Hobson: Harmonic analysis, three hours. — By Dr. H. F. Baker: Theory of functions, three hours; Analysis, three hours. — By Mr. E. T. Whittaker: The differential equations of applied mathematics, three hours. — By Mr. H. W. Richmond: Analytical geometry, three hours. — By Mr. R. A. Herman: Hydrodynamics, two courses. — By Dr. A. N. Whitehead: Symbolic logic and its applications to mathematics. —By Mr. A. Berry: Elliptic functions, three hours. — By Mr. C. T. Bennett: Line geometry, three hours. — By Mr. E. W. Barnes: Linear differential equations, three hours.

Easter term, 1906. — By Professor A. R. Forsyth: Partial differential equations III, three hours. — By Professor J. Larmor: Theory of gases and thermodynamics, two hours. — By Professor J. J. Thomson: Electricity and magnetism, three hours. — By Dr. E. W. Hobson: Theory of the continuum, three hours. — By Dr. H. F. Baker: Theory of functions, three hours; Analysis, three hours. — By Mr. W. L. Mollison: Theory of potential and electrostatics, three hours. — By Dr. A. N. Whitehead: Non-euclidean geometry. — By Mr. A. Berry: Transformation of elliptic functions. — By Mr. G. H. Hardy: Integral functions.

Long vacation, 1906.—By Mr. H. W. Richmond: Geometry, three hours. — By Mr. W. M. Coates: Electricity and magnetism. — By Mr. J. G. Leathem: Physical optics, three hours. — By Mr. A. Young: Theory of invariants, three hours.

PROFESSOR C. J. DE LA VALLÉE-POUSSIN, of the University of Louvain, has been awarded the Belgian decennial prize in mathematics.

DR. K. CARDA, of the University of Vienna, has been appointed associate professor of mathematics at the technical school in Vienna.

DR. M. DEHN, of the University of Münster, has been promoted to an associate professorship of mathematics.

PROFESSOR E. DOLEZAL, of Leoben, has been appointed professor of practical geometry at the technical school in Vienna.

PROFESSOR F. HAHN, of the school of engineering at Lausanne, has been appointed professor of applied mathematics at the University of Nancy.

DR. G. HAMEL, of the technical school at Karlsruhe, has been appointed professor of mechanics at the German technical school in Brünn, Austria.

PROFESSOR C. KOEHLER, of the University of Heidelberg, has been promoted to a full professorship of mathematics.

PROFESSOR A. SOMMERFELD, of the technical school at Aachen, has been decorated with the order of the red eagle of the fourth class.

DR. J. WEINGARTEN, of the technical school at Berlin, has been appointed honorary professor of mathematics at the University of Freiburg.

PROFESSOR W. V. DYCK, of the technical school at Munich, has been decorated with the Prussian order of the crown of the second class.

MR. F. S. PINKERTON has been appointed professor of applied mathematics at the University of South Wales, Cardiff, Wales.

DR. R. BAIRE has been appointed instructor in pure mathematics at the University of Dijon.

DR. P. BOUTROUX has been appointed master of conferences in mathematics at the University of Montpellier.

MR. TH. FRIESSENDORFF has been appointed docent in applied mathematics at the University of St. Petersburg.

DR. R. FUËTER has been appointed docent in mathematics at the University of Marburg.

DR. E. LANDAU, of the University of Berlin, has been promoted to an associate professorship of mathematics at the same institution.

DR. E. PRINGSHEIM, of the University of Berlin, has been appointed professor of theoretical physics at the University of Breslau.

PROFESSOR J. RUIS CASTIZO, of the University of Zaragoza, has been appointed professor of theoretical mechanics at the University of Madrid.

PROFESSOR K. SCHWARZSCHILD, of the University of Göttingen, has been elected to membership in the royal society of Göttingen.

PROFESSOR K. HELLMER, of the technical school at Brünn (Austria) and PROFESSOR A. SCHELL, of the technical school at Vienna, have retired, each with the honorary title of Hofrat.

AT the Paris Ecole polytechnique, Professor G. KOENIGS has been appointed "répétiteur" in analysis; Dr. R. BRICARD and Dr. M. FOUCHÉ to the same position in geometry.

AT the University of Kansas, Professor H. B. NEWSON has been promoted to a full professorship of mathematics, and Mr. J. A. G. SHIRK has been appointed instructor in mathematics.

THE following changes have been announced at the University of Wisconsin: Mr. W. M. PERSONS has resigned his instructorship in mathematics to become assistant professor of statistics at Dartmouth College; Mr. E. R. SMITH, Mr. E. A. MORITZ, Mr. R. S. PEATTER and Dr. D. A. LEHMAN have been appointed instructors in mathematics.

MISS S. E. CRONIN, Mr. J. SUTER, and Mr. G. ANDERSON have been appointed instructors in mathematics at the Iowa State College, Ames, Iowa.

PROFESSOR J. E. BONEBRIGHT, of the Colorado Agricultural College, has been appointed professor of mathematics at Ottawa University, Ottawa, Kansas.

DR. MAX MASON has been promoted to an assistant professorship in mathematics in the Sheffield Scientific School, Yale University.

MR. G. H. LIGHT and Mr. W. A. TEHRING have been appointed instructors in mathematics at Purdue University.

MR. H. W. STAGER and Mr. T. G. BROWN have been appointed assistants in mathematics at Stanford University.

PROFESSOR OTTO STOLZ, of the University of Innsbruck, died November 23, at the age of sixty-three years. Professor VICTOR SCHLEGEL, of the Höhere Maschinenbauschule at Hagen, died November 22 at the age of sixty-three years. Professor Schlegel had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1894.

THE following catalogues of second-hand books on mathematics and physics have recently appeared: A. Hermann, 6 rue de la Sorbonne, Paris V, catalogue no. 86, containing 4073 titles in mathematics; Macmillan and Bowes, 1 Trinity St., Cambridge, England, special sale list of works on applied mathematics; Gustav Fock, Schlossgasse 7, Leipzig, hand apparatus of the late Professor Guido Hauck, 1960 dissertations and reprints; Ernst Carlebach, Heidelberg, catalogue no. 276, 842 entries in science and mathematics; Martinus Nijhoff, the Hague, Nobelstraat 18, catalogue no. 342, containing 1589 titles in exact sciences.

THE library of the late Professor Luigi Cremona, of Rome, is offered for sale. It consists of 3657 reprints and monographs in Italian, 1163 in French, 995 in German, 509 in English and 23 in Latin. The list of books is, however, very small. There are 231 books in Italian, 208 in French, 209 in German, 88 in English, and 18 in Latin. There are also a number of other entries in the catalogue relating to miscellaneous memoirs and periodicals.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMSLER-LAFFON (J.). Zur Lebensgeschichte von Franz Neumann (1798–1895). Zürich, 1904. 8vo. 16 pp. M. 1.00

BAILLAUD (B.). See HERMITE.

BALL (W. W. R.). Histoire des mathématiques. Traduction française, sur la 3e édition anglaise, avec de nombreuses notes et d'importantes additions. Vol. I: Depuis les temps les plus reculés jusqu'à Newton. Paris, Hermann, 1905. 8vo. 425 pp. F. 12.00

Beck (H.). Die Strahlenketten im hyperbolischen Raum. (Diss.) Bonn, 1905. 8vo. 55 pp.

Boegehold (F.). Ableitung einiger Eigenschaften des Kegelschnittbüschels aus orthogonalen Invarianten und Covarianten. Jena, 1905. 8vo. 78 pp. M. 2.00

Bourget (H.). See Hermite.

Bürklen (O. T.). Aufgabensammlung zur analytischen Geometrie der Ebene. Leipzig, Göschen, 1905. 16mo. 196 pp. Cloth. (Sammlung Göschen, No. 256.) M. 0.80

Goursat (E.). Cours d'analyse mathématique. Vol. II: Théorie des fonctions analytiques; équations différentielles; équations aux dérivées partielles; éléments du calcul des variations. Paris, Gauthier-Villars, 1905. 8vo. 6 + 640 pp. (Cours de la Faculté des sciences de Paris.) Fr. 20.00

—— A course in mathematical analysis. Translated by E. R. Hedrick. Vol. I: Derivatives and differentials; definite integrals; expansion in series; applications to geometry. Boston, Ginn [1904]. 8vo. 8 + 548 pp. Cloth. $4.00

Haller (S.). Ueber die Brennpunktskurven zweier besonderer Gruppen von Kegelschnittsystemen. (Progr.) München, 1905. 8vo. 31 pp.

Hartwig (J.). Die Sätze von Pascal und Brianchon auf der Kugelfläche. (Progr.) Wiener-Neustadt, 1905. 8vo. 11 pp.

Heberdey (P.). Die Kegelschnitte. (Progr.) Wien, 1905. 8vo. 35 pp.

Hedrick (E. R.). See Goursat (E.).

Hermite et Stieltjes. Correspondance d'Hermite et de Stieltjes, publiée par les soins de B. Baillaud et H. Bourget, avec une préface de E. Picard. Vol. II: 18 octobre 1889 — 15 décembre 1894. Paris, Gauthier-Villars, 1905. 8vo. 6 + 466 pp. Fr. 16.00

Issaly. La cinématique dans les rapports, absolus avec les pseudo-surfaces, et conditionnels avec les surfaces. Etude servant d'application aux "Principes fondamentaux de la théorie des pseudo-surfaces" du même auteur et de son mémoire "La géométrie non-euclidienne et l'insuffisance de ses principes." Paris, Hermann, 1905. 8vo. 59 pp. F. 3.00

Keyser (C. J.). Concerning certain 4-space quintic configurations of point ranges and congruences, and their sphere analogues in ordinary space. 4to. (*American Journal of Mathematics* 27, pp. 303–314.)

Kommerell (K.). Riemannsche Flächen im ebenen Raum von vier Dimensionen. (Progr.) Heilbronn, 1905. 8vo. 49 pp. M. 1.60

Leonard (H. B.). On the definition of reducible hypercomplex number systems. II. 4to. (*American Journal of Mathematics* 27, pp. 381–405.)

Lorey (W.). Ueber die Wohltat und das Werden der Zahl. (Progr.) Görlitz, 1905. 4to. 10pp.

Mateu (A. F.). Cálculo infinitesimal. Vol. I: Nociones generales abstractas de cálculo diferencial. Oviedo, 1904. 4to. 120 pp. F. 10.00

Neikirk (L. I.). Groups of order p^m which contain cyclic subgroups of order p^{m-3}. 8vo. 65 pp. (*Publications of the University of Pennsylvania*, series in mathematics, No. 3.)

Noth (F.). Differenzialinvarianten und invariante Differenzialgleichungen zweier 10-gliedriger Systeme. Leipzig, 1905. 8vo. 32 pp. M. 1.60

PIERPONT (J.). Lectures on the theory of functions of real variables. Vol. I. Boston, Ginn, 1905. 8vo. 12 + 560 pp. Cloth. $4.50

SCHEFFERS (G.). Lehrbuch der Mathematik für Studierende der Naturwissenschaften und der Technik. Einführung in die Differential- und Integralrechnung und in die analytische Geometrie. Leipzig, Veit, 1905. 8vo. 8 + 682 pp. M. 16.00

SCHMIDL (J.). Ueber die sphärischen Kegelschnitte und ihre orthogonalen Projektionen. (Progr.) Prag-Neustadt, 1905. 8vo. 24 pp.

SEVERI (F.). Complementi di geometria proiettiva. Raccolta di oltre 300 problemi colle relative soluzioni. Bologna, Zanichelli, 1906. 8vo. 6 + 427 pp. L. 10.00

SMITH (D. E.). A portfolio of portraits of eminent mathematicians. Chicago, Open Court Publishing Co., 1905. Portfolio. (Descartes, Pythagoras, Archimedes, Fermat, Leonardo of Pisa, Euclid, Cardan, Leibniz, Napier, Vieta, Newton, Thales.) On Jap. vellum $5.00; on Amer. plate paper, $3.00.

STIELTJES. See HERMITE.

SUPPANTSCHITSCH (R.). Einiges über eine besondere Lage von Flächen zweiter Ordnung. (Progr.) Wien, 1905. 8vo. 20 pp.

TOWNSEND (E. J.). Arzelà's condition for the continuity of a function defined by a series of continuous functions. 8vo. (*Bulletin of the American Mathematical Society* (2) 12, pp. 7–21.)

VITALI (G.). Sul problema della misura dei gruppi di punti di una retta. Nota. Bologna, Gamberini e Parmeggiani, 1905. 8vo. 5 pp.

WICKE (F.). Ueber ultra-Bernoullische und ultra-Eulersche Zahlen und Funktionen und deren Anwendung auf die Summation von Reihen. (Diss.) Jena, 1905. 8vo. 68 pp., 2 plates.

WINTER (A.). Ueber die logarithmischen Grenzfälle der hypergeometrischen Differentialgleichungen mit zwei endlichen singulären Punkten. (Diss.) Kiel, 1905. 8vo. 73 pp.

II. ELEMENTARY MATHEMATICS.

BAKER (A. L.). The art of geometry. A laboratory manual for students' use, to accompany any text-book. Boston, Sibley [1905]. 12mo. 48 pp. Cloth. $0.50

CATANIA (S.). Corso di algebra elementare ad uso delle scuole secondarie superiori. Parte I, per la prima classe liceale, comprese le nozioni di trigonometria, con numerosi esercizi, con le risposte, secondo i nuovi programmi ministeriali. Catania, Giannotta, 1906. 16mo. 8 + 183 pp. L. 2.00

CLAY (C. M.). Examples in algebra. London, 1905. 8vo. Cloth. 4s.

COSTANZO (G.) E NEGRO (C.). Geometria intuitiva e rudimenti di disegno geometrico per le prime classi del ginnasio, a norma delle ultime disposizioni ministeriali. Bologna, Zanichelli, 1906. 16mo. 3 + 141 pp. L. 2.00

CRACKNELL (A. G.). Practical mathematics (stage 1). 3d edition. London and New York, Longmans, 1904. 12mo. 6 + 378 pp. Cloth. $1.10

EHRIG (G.). Ueber Stoff und Methode des mathematischen Unterrichts an Baugewerkschulen. Leipzig, 1905. 8vo. 78 pp. M. 2.00

Grévy (A.). Géométrie théorique et pratique. Paris, Vuibert et Nony, 1905. 16mo. 8 + 472 pp. Fr. 3.50

Gross (H.). See Teichmann (K.).

Hall (H. S.) and Knight (S. R.). Elementary trigonometry. 4th edition, enlarged. London, Macmillan, 1905. 12mo. 432 pp. Cloth. 4s. 6d.

Hamelin (G.). See Naud (L.).

Holzmüller (G.). Die Planimetrie für das Gymnasium, im Anschluss an die preussischen Lehrpläne von 1901 methodisch bearbeitet. Nach Jahrgängen geordnet und mit zahlreichen Uebungsaufgaben zur freien Auswahl versehen. Teil I: Von Quarta bis Untersekunda einschliesslich reichend. 2te Auflage. Leipzig, Teubner, 1905. 8vo. 8 + 240 pp. Cloth. M. 2.40

King's Scholarship questions from 1889 to 1904, with answers in arithmetic and algebra. London, Clive, 1905. 12mo. 186 pp. 6d.

Knight (S. R.). See Hall (H. S.).

London matriculation mathematics papers, from January 1903 to June 1905. London, Clive, 1905. 12mo. 146 pp. Cloth. (University tutorial series.) 1s. 6d.

Meiser (W.). Beweise und Lösungen zu Lehrsätzen und Aufgaben aus der Geometrie. (Progr.) Neustadt a. A., 1905. 8vo. 52 pp.

Müsebeck (C.). Sammlung von Aufgaben aus der Planimetrie, Trigonometrie und Stereometrie, enthaltend Aufgaben zu Teil I, Ausgabe B für Realschulen des Leitfadens der Elementar-Mathematik von H. Lieber und F. von Lühmann. Berlin, Simion, 1905. 8vo. 7 + 115 pp. Cloth. M. 2.00

Naud (L.) et Hamelin (G.). Cours résumé de mathématiques (arithmétique, système métrique, géométrie et algèbre), à l'usage des candidats aux examens de l'administration des postes et des télégraphes. Paris, Courrier des examens [1905]. 8vo. 230 pp. (Bibliothèque du Courrier des examens des postes, des télégraphes et des téléphones.) Fr. 2.80

Negro (C.). See Costanzo (G.).

Pallete (J. G.). Trigonometría rectilínea e esférica. 10a edicion, Madrid, 1905. 4to. 144 pp. F. 5.00

Pedrotti (L.). Risoluzione delle equazioni di 1o e 2o grado. Milano, Sonzogno, 1905. 16mo. 61 pp. (Biblioteca del popolo, No. 376.) L. 0.15

Pokorny (J.). Užiti algebry v kupecké aritmetice. (Progr.) Chrudim, 1905. 8vo. 22 pp.

Rosenberg (K.). Methodisch geordnete Sammlung von Aufgaben aus der Arithmetik und Algebra für Lehrer- und Lehrerinnen- Bildungsanstalten sowie für andere gleichgestellte Lehranstalten. 4te Auflage. Wien, Hölder, 1905. 8vo. 4 + 255 pp. Cloth. M. 2.24.

Russell (J. W.). Elementary treatise on pure geometry, with examples. Revised edition. London, Frowde, 1905. 12mo. 382 pp. Cloth. 9s.

Schmidt (A.). Beiträge zum mathematischen Unterricht: Berechnung der Logarithmen in Untertertia; Einführung der complexen Zahlen. Berlin, 1905. 4to. 16 pp. M. 1.00

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LIBRARY

THE TWELFTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Twelfth Annual Meeting of the Society was held in New York City on Thursday and Friday, December 28–29, 1905. The interest of the occasion was greatly increased by the simultaneous meetings of the American Physical Society and the Astronomical and Astrophysical Societies of America. The large attendance and the extensive programmes gave additional importance to an otherwise noteworthy scientific convention. Community of interest received due attention. A joint session of the Mathematical and Physical Societies was held on Friday afternoon for the purpose of hearing Professor V. F. Bjerknes, of the University of Stockholm, who spoke on "Experimental demonstration of hydrodynamic action at a distance." On Friday evening nearly ninety representatives of the three societies attended a dinner organized in honor of Professor Bjerknes. The common luncheon between each day's sessions afforded an excellent opportunity to renew and make acquaintance and to compare notes scientific or otherwise. Informal gatherings were also held on Thursday evening.

The attendance at the four sessions of the Mathematical Society, which exceeded that of any previous meeting, included the following sixty-six members:

Professor Cleveland Abbe, Professor O. P. Akers, Miss Grace Andrews, Professor G. A. Bliss, Professor E. W. Brown, Dr. W. H. Bussey, Dr. J. E. Clark, Miss Emily Coddington, Professor F. N. Cole, Miss E. B. Cowley, Professor E. S. Crawley, Professor D. R. Curtiss, Dr. W. S. Dennett, President E. A. Engler, Professor T. C. Esty, Professor H. B. Fine, Professor B. F. Finkel, Mr. A. B. Frizell, Professor A. S. Gale, Miss A. B. Gould, Miss Ida Griffiths, Professor F. H. Hodge, Professor E. V. Huntington, Professor J. I. Hutchinson, Mr. S. A. Joffe, Dr. Edward Kasner, Professor O. D. Kellogg, Professor C. J. Keyser, Professor Gustave Legras, Professor E. O. Lovett, Professor James Maclay, Professor H. P. Manning, Professor Max Mason, Professor Helen A. Merrill, Professor Mansfield Merriman, Dr. C. L. E. Moore, Professor G. D. Olds, Professor W. F. Osgood, Dr. J. L. Patterson, Professor Alexander Pell, Professor A. W. Phillips, Professor James Pierpont, Mr. R. G. D. Richardson, Dr. F. H. Safford, Professor Arthur Schultze, Professor Charlotte A. Scott, Dr. C. H. Sisam, Dr. Clara E. Smith, Professor P. F. Smith, Dr. H. F. Stecker, Professor W. E. Story, Dr. C. E. Stromquist, Professor H. D. Thompson, Miss M. E. Trueblood, Professor H. W. Tyler, Miss A. L. Van Benschoten, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor J. B. Webb, Professor A. G. Webster, Mr. W. D. A. Westfall, Professor H. S. White, Dr. E. B. Wilson, Mr. J. E. Wright.

The President of the Society, Professor W. F. Osgood, occupied the chair, being relieved at the Friday afternoon session by Professor E. W. Brown. President Carl Barus, of the Physical Society, presided at the joint session. The Council announced the election of the following persons to membership in the Society: Mr. R. L. Börger, University of Missouri; Professor W. B. Carver, Ursinus College; Mr. A. J. Champreux, University of California; Dr. Emily Coddington, New York, N. Y.; Dr. F. J. Dohmen, University of Texas; Professor O. E. Glenn, Drury College; Mr. E. S. Haynes, University of Missouri; Professor J. H. Jeans, Princeton University; Mr. A. R. Maxson, Columbia University; Professor J. F. Travis, Georgia School of Technology; Professor Vito Volterra, University of Rome; Miss May E. G. Waddell, Orono, Canada. Nineteen applications for admission to membership in the Society were received.

Reports were received from the Treasurer, Librarian, and Auditing Committee. These reports will appear in the Annual Register, now in press. The membership of the Society has increased during the past year from 473 to 512. The number of papers presented at all meetings during the year was 147, as against 148 in 1904. The total attendance of members was 280; 161 members attended at least one meeting during the year. The library now contains about 2,000 bound volumes; a list of the journals and of the accessions during 1905 is printed in the Annual Register. The Treasurer's report shows a balance of $3,833.01 on hand December 16, 1905; of this balance $2,132.58 is credited to the life-membership fund.

At the annual election, which closed on Friday morning, the following officers and other members of the Council were chosen:

Vice-Presidents, Professor Charlotte A. Scott,
Professor Irving Stringham,
Secretary, Professor F. N. Cole,
Treasurer, Dr. W. S. Dennett,
Librarian, Professor D. E. Smith,

Committee of Publication,

Professor F. N. Cole,
Professor Alexander Ziwet,
Professor D. E. Smith.

Members of the Council to serve until December, 1908,

Professor C. L. Bouton,
Professor L. E. Dickson,
Dr. Edward Kasner,
Professor E. J. Townsend.

The following papers were read at the meeting:

(1) Mr. R. G. D. Richardson: "Multiple improper integrals."

(2) Mr. A. B. Frizell: "On the continuum problem" (preliminary communication).

(3) Professor D. R. Curtiss: "The vanishing of the wronskian and the problem of linear dependence."

(4) Professor J. I. Hutchinson: "On certain automorphic groups whose coefficients are integers in a quadratic field."

(5) Professor E. V. Huntington: "Note on the fundamental propositions of algebra" (preliminary communication).

(6) Professor C. J. Keyser: "Concerning a self-reciprocal plane geometry."

(7) Dr. C. L. E. Moore: "Geometry of circles orthogonal to a given sphere."

(8) Dr. Edward Kasner: "Invariants of differential elements for arbitrary point transformation."

(9) Mr. A. B. Frizell: "A method of building up the fundamental operation groups of arithmetic."

(10) Professor G. A. Bliss: "A proof of the fundamental theorem of analysis situs."

(11) Professor O. P. Akers: "On the congruence of axes in a bundle of linear line complexes."

(12) Dr. Peter Field: "Note on certain groups of transformations of the plane into itself."

(13) Mr. **George Peirce**: "A new approximate construction for π."

(14) Professor **Max Mason**: "Curves of minimum moment of inertia."

(15) Professor **A. G. Webster**: "Application of a definite integral involving Bessel's functions to the self-inductance of solenoids."

(16) Mr. **J. E. Wright**: "Correspondences and the theory of continuous groups."

(17) Mr. **J. E. Wright**: "An application of the differential invariants of space."

(18) Dr. **Clara E. Smith**: "A theorem of Abel and its application to the development of an arbitrary function in terms of Bessel's functions."

(19) Professor **V. F. Bjerknes**: "Experimental demonstration of hydrodynamic action at a distance."

(20) Dr. **R. P. Stephens**: "On the pentadeltoid."

(21) Professor **M. I. Pupin**: "Establishment of the steady state in a sectional wave conductor."

(22) Dr. **J. J. Quinn**: "A linkage for the kinematic description of a cissoid."

Dr. Stephens was introduced by Professor Morley. Mr. Peirce's paper was communicated to the Society through Professor E. W. Brown, Dr. Quinn's paper through Dr. A. L. Baker. In the absence of the authors Mr. Peirce's paper was read by Professor Brown, and the papers of Dr. Field, Professor Pupin and Dr. Quinn were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The method used by Mr. Richardson for defining multiple improper integrals differs from that of Professor Pierpont (*Transactions*, January, 1906). Analogous to the definition given by Vallée Poussin (*Journal de Mathématiques*, series 4, volume 8, 1892) for one variable, a truncated limited function $f_{\lambda_1\lambda_2}$ of m variables $x_1, \cdots, x_m$ is defined over a limited aggregate $\mathfrak{A}$. The integral $I_{\lambda_1\lambda_2}$ of this truncated function always exists, and if $\underset{\lambda_1=\infty\,\lambda_2=\infty}{\text{limit}} I_{\lambda_1\lambda_2}$ exists, it is called the improper integral of f in $\mathfrak{A}$. The integrals defined by this method are absolutely convergent. Mr. Richardson has shown that the integrals defined by Professor Pierpont are also absolutely convergent and that the two integrals are identical. Various properties, theorems of the

mean, and convergent tests are derived from this general integral. If the variables are broken up into two sets $x_1, \cdots, x_n$ and $x_{n+1}, \cdots, x_m$, and if $\mathfrak{B}$ and $\mathfrak{C}$ are the n-way and $(m-n)$-way aggregates corresponding to $\mathfrak{A}$, we may call

$$J(x_1, \cdots, x_n) = \int_{\mathfrak{C}} f(x_1, \cdots, x_m)$$

an integral depending on the parameters $x_1, \cdots, x_n$, and investigate it with regard to continuity, differentiation, and integration. If J is an integral uniformly convergent in $\mathfrak{B}$, it is a continuous function of $x_1, \cdots, x_n$. If the integral of f over $\mathfrak{C}$ is regularly convergent, and the integral of f over $\mathfrak{B}$ is uniformly convergent we may write

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f(x_1, \cdots, x_m) = \int_{\mathfrak{C}}\int_{\mathfrak{B}} f(x_1, \cdots, x_m).$$

Various other criteria for differentiating under the integral sign and for reduction of multiple integrals are developed.

2. Mathematicians have naturally looked to the problems of counting the points of the linear continuum or of arranging them in a well ordered set for applications of Cantor's transfinite numbers, and perhaps the most general opinion is that the number of points on a segment of a straight line will turn out to be either Aleph-eins or some subsequent Aleph. Mr. Frizell's paper adopts $\aleph_0^{\aleph_0}$ for the cardinal number of the continuum and considers the possibility of this expression being equal to Aleph-null. This leads to a scrutiny of the definition of involution for transfinite numbers, in view of the remarkable relations

$$\aleph_0 = \aleph_0^2 = \aleph_0^3 = \cdots = \aleph_0^\nu,$$

$$2^{\aleph_0} = 3^{\aleph_0} = \cdots = \mu^{\aleph_0} = \cdots = \aleph_0^{\aleph_0}.$$

The writer's opinion, however, is that no satisfactory solution is likely to be reached except by exhibiting the continuum as a well ordered class possessing a definite rank in Cantor's series of ordinal types. A method of doing this is suggested by first well ordering the class of all finite simple continued fractions and then ordering all possible infinite sequences composed of members of this class according to the place in the scheme of the elements omitted in their formation. The writer hopes to develop this process more fully in a subsequent communication.

3. Although the vanishing of the wronskian is both a necessary and sufficient condition for the linear dependence of n analytic functions $u_1, u_2, \cdots, u_n$, this is by no means always the case for sets of functions of a real variable which are subject to no other restriction than that they have finite derivatives of order $> n - 2$ in the interval under consideration. In Professor Curtiss's paper a theorem concerning the vanishing of the wronskian in an infinite set of points is established, and by its aid new criteria for linear dependence are obtained.

4. The groups considered by Professor Hutchinson are formed by the linear unimodular transformations of a single variable, the coefficients being of the form $\alpha + \lambda\alpha'$ in which λ is a root of the equation $\lambda^2 - m\lambda + n = 0$, and α, α', m, n are integers. Linear relations among the coefficients are assumed so as to reduce the number of arbitrary integers $\alpha, \alpha', \cdots$ involved in the coefficients to four, among which a quadratic relation exists on account of the determinant of the transformation being unity. The main object of the paper is to determine the forms of these relations so that the transformations shall form a group properly discontinuous in the plane of the transformed variable.

5. In Professor Huntington's paper the attempt is made to present sets of postulates for the various branches of real algebra in a form available for pedagogical purposes. For example, the fundamental propositions selected for the algebra of all real numbers are the following, the operations of addition and multiplication and the relation of order being taken as the fundamental concepts:

*A*1. Every two elements a and b determine uniquely an element $a + b$, called their sum.

*A*2. $(a + b) + c = a + (b + c)$.

*A*3. $a + b = b + a$.

*A*4. If $a + x = a + y$, then $x = y$.

*A*5. There is an element z such that $z + z = z$. This element z proves to be uniquely determined, and is called 0.

*A*6. For every element a there is an element a' such that $a + a' = 0$.

$M1$. Every two elements a and b determine uniquely an element ab, called their product; and if $a \neq 0$ and $b \neq 0$, then $ab \neq 0$.*

$M2$. $(ab)c = a(bc)$.

$M3$. $ab = ba$.

$M4$. If $ax = ay$, and $a \neq 0$, then $x = y$.

$M5$. There is an element u, different from 0, such that $uu = u$. This element u proves to be uniquely determined, and is called 1.

$M6$. For every element a, not 0, there is an element a'' such that $aa'' = 1$.

$AM1$. $a(b + c) = ab + ac$.

$S1$. If $a \neq b$, then either $a < b$ or $b < a$.

$S2$. If $a < b$, then $a \neq b$.

$S3$. If $a < b$ and $b < c$, then $a < c$.

$S4$. Dedekind's postulate.

$SA1$. If $x < y$ then $a + x < a + y$.

$SM1$. If $a > 0$ and $b > 0$, then $ab > 0$.

The sets of postulates for the algebra of positive reals (with or without 0), the algebra of all integers, the algebra of positive integers (with or without 0), etc., are obtained by replacing one or more of the postulates $A5$, $A6$, $M5$, $M6$ by their opposites, and slightly modifying $SA1$ and $SM1$.

6. Of the infinitely many possible geometries of a given analytic type, all but a few deal with configurations too remote from (present) intuition to engage the interest and thereby to justify or invite detailed investigation and exposition. Accordingly it is a matter of judgment to select from possible geometries of a type those that are worthy of such attention. That determination once effected, or even in course of it, interest attaches chiefly to the general comparative anatomy of the allied theories rather than to a minute or histological study of a single one of the variety. Professor Keyser's paper adds another theory to the list of the noteworthy doctrines of the type of the Plücker line geometry. It is a plane geometry in which the circle range (one parameter circle system determined by two circles) is employed as element. The self-reciprocal character of the geometry is evident in the fact that the element is equally determinable as the intersection of two circle congruences,

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* The latter part of $M1$ is redundant, in view of $A4$ and $AM1$.

where by congruence is meant the system of circles orthogonal to a given circle. The circle and the circle congruence are reciprocal elements in the circle geometry of the plane. The circle range can be regarded either as a linear "locus" of circles or as a linear "envelope" of a pencil of circle congruences. The circles and circle congruences are related as pole and polar with respect to the invariant circle configuration composed of the point circles of the plane.

7. If p, p' are the point spheres of the pencil determined by the fixed sphere S and the variable plane P, as P rotates about a line l, a circle c orthogonal to S will be traced by the points p, p'. By means of this transformation Dr. Moore discusses the geometry of circles orthogonal to a given sphere.

8. The simplest example of an expression which is invariant with respect to arbitrary point transformation is the anharmonic ratio of the tangents to four curves passing through a common point. The invariance here results from the fact that lineal elements with a common point undergo homographic transformation. Dr. Kasner determines all invariants depending upon the curvature as well as the direction of any number of curves passing through a point. (The results obtained do not agree with those given by Rabut in a memoir published in the *Journal de l'Ecole Polytechnique* for 1898). The eight-parameter group induced on a bundle of curvature elements is studied in detail. A representation in space of four dimensions suggests a classification of differential equations of the second order according to *rank* — a classification which is invariant with respect to point transformation. Equations of the first rank have been studied by Lie and R. Liouville, but the others have apparently not been noticed. The group of all conformal transformations is found to induce a four-parameter group; it leads to absolute invariants not always expressible in terms of angles.

9. The problem of establishing the laws of algebra on a logical basis has two aspects. One is their reduction to a working system of postulates; this is discussed in Professor Huntington's paper. The other, which forms the subject of Mr. Frizell's second paper is that of providing a means for readily obtaining the proofs. This is effected chiefly by aid of the following proposition in the abstract theory of groups. Let

C denote a class whose members admit of comparison as equal or not equal, and suppose that C constitutes an abelian semigroup with respect to a certain rule of combination denoted by $\circ$. Form the class $K = \{(m, n)\}$ of pairs of members of C. In K, declare $(m, q) = (n, p)$ if $m \circ p = n \circ q$ and lay down a rule $\odot$ whereby $(m, q) \odot (n, r) = (m \circ n, q \circ r)$. Then it can be shown that K is a C-class, $K = C_1$, and that $\odot$ denotes a rule of combination for C_1 such that C_1 forms a group with respect to $\odot$. By three applications of this theorem we obtain from the class of natural numbers $C = \{n\}$ first the class of absolute rational numbers $C_1 = \{R\}$, then the whole class of absolute numbers $C_2 = \{X\}$, and finally the class of complex numbers $C_3 = \{Z\}$ not including zero. The process consists in representing the successive classes geometrically in the first cases by the circumferences of a set of concentric circles, in the last case by points in a plane. The method admits of extension to the development of the number system of quaternions.

10. Professor Bliss gave a proof of the theorem that a continuous closed curve without double points and with a continuously turning tangent divides the plane into two regions, an interior and an exterior. The proofs heretofore given of the division of the plane by a Jordan curve have been of two kinds, those in which only the continuity of the curve has been assumed, and those in which generality in the assumption has to some extent been sacrificed in order to attain greater simplicity of proof. The present proof is of the second kind.

11. In Professor Akers's paper the lines of the (3, 2) congruence, made up of the axes in the bundle of complexes $\sigma_1(p_{23} - e_1 p_{41}) + \sigma_2(p_{31} - e_2 p_{42}) + \sigma_3(p_{12} - e_3 p_{43}) = 0$, were arranged on ∞^1 ruled surfaces of order five. The σ_i are regarded as point coördinates in the plane at infinity; and in this way the configuration is mapped on the plane. The points σ of the plane at infinity and the lines of the congruence are in (1, 1) correspondence. The condition was obtained under which the complex to which the surfaces belong becomes special, and all the exceptional lines were determined. The six congruences having the same focal surface were discussed and compared.

12. Dr. Field's note deals with certain groups of transformations of the plane into itself, which were suggested in connec-

tion with the study of a plane quintic curve with five cusps. The paper is published in full in the present number of the BULLETIN.

13. Mr. George Pierce's paper contained an approximate construction for π which gives the value of this constant with an error of .000048. It compares very favorably with all previous constructions as to both simplicity and accuracy.

14. Professor Mason discussed the problem of determining a plane curve joining two given points whose moment of inertia with respect to a given point or a given line is a minimum. In the first case the extremals are expressed by trigonometric functions, in the second by elliptic functions. Both problems possess discontinuous solutions for certain positions of the given end points.

15. In Professor Webster's paper the solenoid is treated as a cylindrical current sheet, and the self-inductance found from that of an infinite sheet by determining the correction for the ends. For this purpose use is made of a result of H. Weber, that the potential due to a uniform circular disk of radius a at a point whose coördinates are z, x is

$$U = 2\pi a \int_0^\infty \frac{e^{-\lambda z}}{\lambda} J_0(\lambda x) J_1(\lambda a)\, d\lambda.$$

From this the axial field is obtained by differentiating with respect to z, the flux by integrating with respect to x, introducing $J_1(\lambda a)$ a second time as a factor. The integration with respect to λ and z may then be carried out after changing the order of integration, giving a convergent series in powers of a/l, where l is the length of cylinder.

16. Mr. Wright pointed out that the basis of a correspondence between two sets of geometric loci must lie in the theory of continuous groups, and he gives some results of a general nature. As a particular application he considered correspondences arising out of the projective group taken together with the straight line in space of three dimensions. He thus obtained theorems that certain conformal and projective groups are simply isomorphic. He also developed, from the group

point of view, the correspondence between three-dimensional lines and four-dimensional points, that between screws and hyperspheres, and that between five-dimensional points and three-dimensional screws.

17. If a family of surfaces $u(x, y, z) = \text{const.}$ can form part of a triply orthogonal system, they must satisfy a differential equation of the third order, $\triangle = 0$. $\triangle$ must be a differential invariant, and hence it is an algebraic invariant of certain known forms. Mr. Wright shows that it is an algebraic invariant of one linear and three quadratic forms in three variables, and its vanishing is equivalent to the geometric condition that the line meet the conics in six points in involution.

18. The formula

$$\frac{2u}{\pi}\int_0^1 dt\int_0^1 \frac{F'(\lambda ut)\lambda d\lambda}{\sqrt{1-\lambda^2}\sqrt{1-t^2}} = F(u) - F(0),$$

first given by Abel, has hitherto been proved only under quite narrow restrictions. Miss Smith's proof founded on the properties of double integrals is valid under very broad conditions. The formula is applied to an investigation of the condition under which the development of an arbitrary function in the form

$$f(x) = \sum_{u=0}^{u=\infty} a_u J_0(ux)$$

is valid. This development, outlined by Schlömilch in 1857, seems, up to the present, to lack any rigorous demonstration of its validity, even under conditions far more stringent than here required.

19. Professor Bjerknes showed how the extensive analogy existing between certain hydrodynamic phenomena and those of electric or magnetic fields could be developed from elementary principles, the principle of continuity and the principle of kinetic buoyancy. This last principle as well as the results relating to the analogy just mentioned were illustrated by a series of experiments.

20. Dr. Stephens's paper will be published in full in the *Transactions*.

21. Professor Pupin's paper discusses the quasi-discontinuous variations of the electric current in a sectional conductor when a steady electromotive force is impressed upon it. Thomson's mathematical discussion of the electric cable problem in 1855 and Kirchhoff's solution in 1858 of Thomson's problem in more general form are special cases of that here treated. The paper contributes an addition to the exceedingly small number of existing illustrations of the application of the Lagrangian method of solving a certain class of differential equations by a finite series of harmonics, the extension of which led to the discovery of Fourier's series. The solution discussed in the paper seems to offer a field for physical applications.

22. Dr. Quinn presented two preliminary theorems on properties of the cissoid, which suggest a mode of constructing a linkage for its kinematic description.

F. N. COLE,
Secretary.

NOTE ON CERTAIN GROUPS OF TRANSFORMATIONS OF THE PLANE INTO ITSELF.

BY DR. PETER FIELD.

(Read before the American Mathematical Society, December 29, 1905.)

IN the study of a plane curve or other configuration determined by five points,* four of the points may be taken as fixed and all the curves can be obtained by taking the fifth as any point in a given one of the one hundred and twenty regions determined in Professor Slaught's thesis.† Professor Slaught's diagram applies to the case of five real points. It is the purpose of this note to indicate diagrams corresponding to the case when four or two of the points are imaginary.

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* Del Pezzo, *Rendiconti Accad. scienze fisiche matem.*, Napoli, ser. 2, vol. 3 (1889), pp. 46–49. Field, *Transactions Amer. Math., Society*, January, 1906.

† "The cross ratio group of one hundred and twenty quadratic Cremona transformations of the plane," *Amer. Journ. of Math.*, vols. 22 and 23. Also see Moore, "The cross ratio group of $n!$ Cremona transformations of order $n-3$ in flat space of $n-3$ dimensions," *Amer. Journ. of Math.*, vol. 22, No. 3, pp. 279-291, and Kantor, Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene (Berlin, 1895).

1. *Four Imaginary Points.*

Let the points be $a \equiv (i:0:1)$, $b \equiv (-i:0:1$, $c \equiv (0:i:1)$, $d \equiv (0:-i:1)$, the fifth point being any real point $x_1:y_1:z_1$. It is clear that the fifth point can not be projected into either of the first four, while the different ways in which the first four can be projected into themselves are given by the transitive group of order 8 and degree 4 [the generators of the group might be taken as the substitutions $(adbc)$ and (ab)]. If $z=0$ be the line at infinity, the fifth point may be taken as any point in one of the eight regions bounded by the lines $x=0$, $y=0$, $x-y=0$, $x+y=0$. This is evident when it is noticed that the generators of the group of transformations corresponding to the preceding substitution are $x':y':z' = y_1:-x_1:z_1$ and $x':y':z' = -x_1:y_1:z_1$.

2. *Two Imaginary Points.*

In case but two of the points are imaginary let the five points be taken as follows: $g_1 \equiv (0:0:1)$, $h_1 \equiv (1:1:1)$, $c \equiv (i:1:0)$, $d \equiv (-i:1:0)$, $e \equiv (x_1:y_1:z_1)$. Four of the preceding five points can be projected into the first four in twelve different ways, because any one of the three real points may be projected into g_1 and either of the two remaining into h_1, while c and d may remain invariant or be interchanged. To project (g_1, h_1, d, c), (g_1, e_1, c, d), (h_1, g_1, c, d) into (g_1, h_1, c, d) we have the transformations

$$x':y':z' = y:x:z, \qquad x':y':z' = (x_1+y_1)x-(x_1-y_1)y:$$

$$(x_1-y_1)x+(x_1+y_1)y:\frac{x_1^2+y_1^2}{z_1}z, \quad x':y':z' = z-x:z-y:z.$$

For these three cases the coördinates of the new fifth point in terms of x_1, y_1, z_1, are respectively

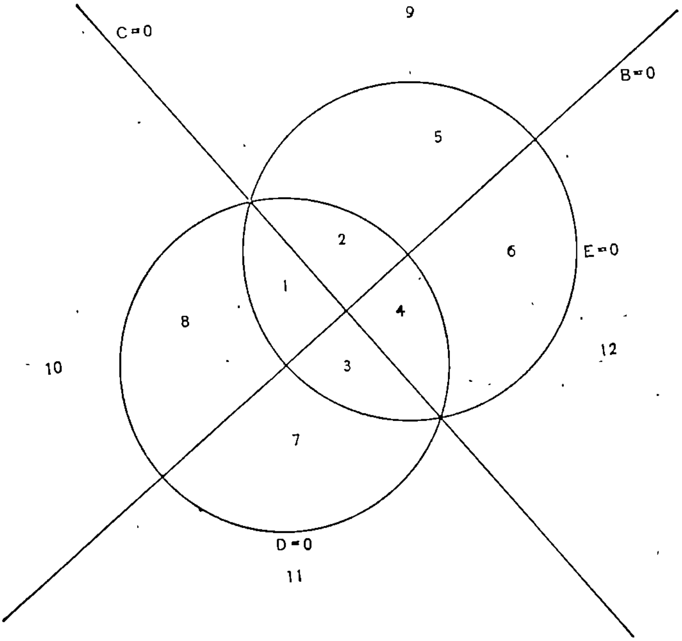
$$x':y':z' = y_1:x_1:z_1, \tag{1}$$

$$x':y':z' = 2y_1z_1:2x_1z_1:x_1^2+y_1^2, \tag{2}$$

$$x':y':z' = z_1-x_1:z_1-y_1:z_1. \tag{3}$$

These last transformations generate a group of order 12. The boundaries of the regions are the curves $A \equiv z = 0$,

$B \equiv x - y = 0$, $C \equiv x + y - z = 0$, $D \equiv x^2 + y^2 - 2z^2 = 0$, $E \equiv (x - z)^2 + (y - z)^2 - 2z^2 = 0$. The way in which the



regions are permuted under the transformations (1), (2), (3) is shown in the following table:

(1)	(13), (24), (56), (78), (9 12), (10 11),
(2)	(1 12), (26), (39), (45), (7 10), (8 11),
(3)	(14), (23), (57), (68), (9 11), (10 12),

In the figure above the line A is taken as the line at infinity.

UNIVERSITY OF MICHIGAN,
November, 1905.

THE MERAN MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held, in affiliation with the seventy-seventh regular convention of the Deutsche Naturforscher und Aerzte, at Meran, in the Tyrol, September 24–30, 1905. Eighty persons attended the sessions, which were presided over by Professors Köhler, Stäckel, Czuber, Geiser and Hensel respectively. The following papers were read:

(1) Professor CZUBER, Vienna: "The question of introducing the calculus into the preparatory schools, from an Austrian standpoint (Report)."

(2) Professor HOČEVAR, Graz: "Should the calculus be introduced into the preparatory schools, or not?"

(3) Professor WÄLSCH, Brünn: "Geometric-mechanic pictures of a new binary-invariant writing of chemical formulas."

(4) Professor HERZ, Vienna: "The year of Christ's birth."

(5) Professor WIEN, Würzburg: "Report on the partial differential equations of physics."

(6) Professor HASENÖHRL, Vienna: "On methods of integration of Maxwell equations for electric vibrations."

(7) Professor GANS, Tübingen: "Gravitation and electromagnetism."

(8) Professor WIRTINGER, Vienna: "On branching of functions of two variables."

(9) Dr. KOEBE, Berlin: "On conformal representation of multiply connected regions, particularly of regions that are bounded by arcs of circles."

(10) Professor LEVI-CIVITA, Padua: "On a technical problem in relation to conformal representation."

(11) Professor SCHOENFLIES, Königsberg: "On the so-called paradoxes of the theory of aggregates."

(12) Professor HENSEL, Marburg: "On the arithmetic properties of algebraic and of transcendental numbers."

(13) Professor ZINDLER, Innsbruck: "Report on the development of differential line geometry."

(14) Professor EPSTEIN, Strassburg: "Report on new investigations of the Riemann ζ-function and its extensions."

(15) Professor KOHN, Vienna: "On doubly apolar quadrics."

(16) Professor SCHLESINGER, Clausenburg: "On a representation of the system of absolute geometry."

(17) Dr. GRÜNWALD, Vienna : "Certain geometric applications of dual numbers."

(18) Professor MÜLLER, Vienna : "Contributions to cyclography."

(19) Professor MÜLLER, Vienna : "Descriptive geometry as an illustration of abstract projective geometry."

(20) Professor DOKULIL, Vienna : "Photogrammetry in the service of the history of art."

(21) Professor DOKULIL, Vienna : "Construction and study of stereoscopic pictures."

(22) Dr. FEJÉR, Clausenburg : "On equilibrium in a resisting medium."

(23) Professor STÄCKEL, Hanover : "Pairs of isometric surfaces."

Abstracts of the papers, so far as obtainable, follow. They are numbered to correspond to the titles in the list above.

1. Austrian schools give more time to mathematics and the sciences than is allowed in Germany, but Professor Czuber urges that the principles of the theory of functions be introduced empirically much earlier than at present. The idea of a derivative and of a definite integral could be taught in the gymnasia without lengthening the time devoted to mathematics. In conclusion he stated exactly formulated proposals as to the manner of carrying out the reform. The hearers, with one exception, agreed heartily with Professor Czuber's views.

2. Professor Hoĉevar answers his question in the affirmative and further discusses the plan outlined in 1.

5. This report will appear in full in the *Jahresbericht.*

6. Professor Hasenöhrl's paper shows how to apply certain methods of the theory of potential and related subjects to the integration of Maxwell's equations.

7. Professor Gans developed the Lorentz idea, that electrons of the same sign repel each other rather less strongly than those of opposite sign attract each other. A conductor may give no insulation for gravitation, yet furnish complete insulation for electrostatic forces. The deviation of this law of attraction from Newton's law depends only on quantities of the second

order. The consequences of this deviation should be investigated.

8. After giving a detailed discussion of the branching of the function $z^3 + 3xz + 2y = 0$ at $x = 0$, $y = 0$, Professor Wirtinger offered various considerations of the branching of a Riemann space, and interpreted the results by analysis situs.

9. Dr. Koebe proved the following theorem: If τ, τ' be two smooth plane regions, each bounded by $\rho + 1$ circles, of which no two cut or touch each other, and if τ can be mapped conformally upon τ', then the analytic function by means of which the representation is effected must be an integral or fractional linear function, and extended the result to apply to certain Riemann surfaces.

10. A problem in technical electricity was shown to depend upon the conformal representation of a multiply connected region bounded by circles. That discussed by Professor Levi-Civita was the determination of the most advantageous arrangement of the auxiliary conductor in the Wurtz lightning conductor.

11. This paper is to appear in the *Jahresbericht.*

13. The full report of which the present paper forms a part will be published in the *Jahresbericht.*

14. After speaking of the properties of Riemann's zeta functions, Dr. Epstein divided the generalizations into two groups, first, that of Dedekind, investigated by Landau, which relates to algebraic numbers, and second that studied by Lipschitz, Hurwitz and others, which bears the same relation to Riemann's zeta function as theta functions of p variables do to elliptic theta functions.

15. Professor Kohn showed that two quadric surfaces would be doubly apolar when one contains two generators of the same system which are polar reciprocal as to the other. Each surface has then a pair of lines reciprocal as to the other, in each generation. The quadrilaterals formed by these generators were then investigated.

16. The subject matter of Professor Schlesinger's paper will be incorporated in his forthcoming book on absolute geometry.

17. Professor Grünwald applies the numbers $\mu + \epsilon\nu$ ($\epsilon^2 = 0$) to the study of directed lines in space.

23. From every pair of applicable surfaces it is possible to derive twenty-eight pairs of applicable surfaces. Professor Stäckel's paper appeared in full in the *Jahresbericht*, volume 14, pages 507–516.

E. A. MILLER,
ELIJAH SWIFT.

THE PRESENT AND THE FUTURE OF MATHEMATICAL PHYSICS.*

ADDRESS DELIVERED BEFORE THE SECTION OF APPLIED MATHEMATICS OF THE INTERNATIONAL CONGRESS OF ARTS AND SCIENCE, ST. LOUIS, SEPTEMBER 24, 1904.

BY PROFESSOR H. POINCARÉ.

WHAT is the present state of mathematical physics? What are its problems? What is its future? Is it about to change its orientation? Will the object and methods of this science appear in ten years to our immediate successors in the same light as they appear to us? Or are we to witness a far-reaching transformation? These are the questions we are forced to face to-day at the outset of our inquiry.

It is easy to ask; difficult to answer. If we felt tempted to hazard a prediction, we should easily resist this temptation by stopping to think of the nonsense the most eminent scholars of a hundred years ago would have spoken in answer to the question of what this science would be in the nineteenth century. They would have thought themselves bold in their predictions; and after the event how timid we should have found them! Do not expect of me therefore any kind of prophesy.

But if, like all prudent physicians, I refuse to give a prognosis, still I cannot deny myself a little diagnosis. Well, then, yes; there are symptoms of a serious crisis, which would seem to indicate that we may expect presently a transformation. However, there is no cause for great anxiety. We are assured that the patient will not die, and indeed we may hope that this crisis will be salutary, since the history of the past would seem to insure that. In fact, this crisis is not the first, and in order to understand it it is well to recall those which have gone before. Allow me a brief historical sketch.

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* Translated with the author's permission by Professor J. W. YOUNG.

Mathematical physics, as we are well aware, is an offspring of celestial mechanics, which gave it birth at the end of the eighteenth century at the moment when it had itself attained its complete development. The child, especially during its first years, showed a striking resemblance to its mother.

The astronomical universe consists of masses, undoubtedly of great magnitude, but separated by such immense distances that they appear to us as material points; these points attract each other in the inverse ratio of the squares of their distances, and this attraction is the only force which affects their motion. But if our senses were keen enough to show us all the details of the bodies which the physicist studies, the spectacle thus disclosed would hardly differ from the one which the astronomer contemplates. There too we should see material points separated by intervals which are enormous in comparison with their dimensions, and describing orbits according to regular laws. These infinitesimal stars are the atoms. Like the stars proper, they attract each other or repel, and this attraction or repulsion, which is along the line joining them, depends only on the distance. The law according to which this force varies with the distance is perhaps not the law of Newton, but it is analogous thereto: instead of the exponent -2 we probably have another exponent, and from this diversity in the exponents proceeds all the diversity of the physical phenomena, the variety in qualities and sensations, all the world of color and sound which surrounds us; in a word, all nature.

Such is the primitive conception in its utmost purity. Nothing remains but to inquire in the different cases, what value must be given to this exponent in order to account for all the facts. On this model, for example, Laplace constructed his beautiful theory of capillarity; he simply regards the latter as a special case of attraction, or, as he says, of universal gravitation, and no one is surprised to find it in the middle of one of the five volumes of his celestial mechanics. More recently Briot believes he has laid bare the last secret of optics, when he has proved that the atoms of the ether attract each other in the inverse sixth power of the distance; and does not Maxwell, Maxwell himself, say somewhere that the atoms of a gas repel each other in the inverse ratio of the fifth power of the distance? We have the exponent -6 or -5, instead of the exponent -2; but it is always an exponent.

Among the theories of this period there is a single one that forms an exception, namely that of Fourier; here there are indeed atoms acting at a distance; they send each other heat, but they do not attract each other, they do not stir. From this point of view, Fourier's theory must have appeared imperfect and provisional to the eyes of his contemporaries, and even to his own.

This conception was not without greatness; it was alluring, and many of us have not definitely given it up; they know that the ultimate elements of things will not be attained, except by disentangling with patience the complex skein furnished us by our senses; that progress should be made step by step without neglecting any intermediate portions; that our fathers were unwise in not wishing to stop at all the stations; but they believe that when we once arrive at these ultimate elements, we shall meet again the majestic simplicity of celestial mechanics.

Nor has this conception been useless; it has rendered us a priceless service inasmuch as it has contributed to making more precise the fundamental concept of the physical law. Let me explain: What did the ancients understand by a law? It was to them an internal harmony, statical as it were, and unchangeable; or else a model which nature tried to imitate. To us a law is no longer that at all; it is a constant relation between the phenomenon of to-day and that of to-morrow; in a word, it is a differential equation.

Here we have the ideal form of the physical law; and, indeed, it is Newton's law which first gave it this form. If, later on, this form has become inured in physics, it has become so precisely by copying as far as possible this law of Newton, by using celestial mechanics as a model.

Nevertheless there came a day when the conception of central forces appeared no longer to suffice, and this is the first of the crises to which I referred a moment ago.

What was done? Abandoned was the thought of exploring the details of the universe, of isolating the parts of this vast mechanism, of analyzing one by one the forces which set them going; and one was content to take as guides certain general principles which have precisely the object of relieving us of this minute study. How is this possible? Suppose we have before us any kind of machine; the part of the mechanism where the power is applied and the ultimate resultant motion alone are visible, while the transmissions, the intermediate gearing whereby the motion is communicated from one part to another, are hidden in the interior and escape our notice; we know not whether the transmissions are made by cog-wheels or by belts, by connecting-rods or other contrivances. Shall we say that it is impossible for us to learn anything about this machine unless we are allowed to take it apart? You well know that such is not the case, and that the principle of the conservation of energy suffices to furnish us the most interesting feature. We can easily show that the last wheel turns ten times more slowly than the first, since these two wheels are visible; and we can conclude therefrom that a couple applied to the first will be in equilibrium with a couple ten times as great applied to the second. To obtain this result, it is in no wise necessary to look into the mechanism of this equilibrium, or to know how the forces balance in the interior of the machine; it is sufficient to make sure that it is impossible for this balancing not to take place.

Very well! In the case of the universe, the principle of the conservation of energy can render us the same service. This universe also is a machine, much more complicated than any in use in the industries, of which nearly all the parts are deeply hidden; but by observing the motion of those which we can see, we can by the aid of this principle draw conclusions which will remain valid no matter what the details of the invisible mechanism which actuates them.

The principle of the conservation of energy, or Mayer's principle, is certainly the most important, but it is not the only one; there are others from which we can derive the same advantage. These are:

Carnot's principle, or the principle of the dissipation of energy.

Newton's principle, or the principle of the equality of action and reaction.

The principle of relativity, according to which the laws of physical phenomena must be the same for a stationary observer as for one carried along in a uniform motion of translation, so that we have no means, and can have none, of determining whether or no we are being carried along in such a motion.

The principle of the conservation of mass, or Lavoisier's principle. I will add the principle of least action.

The application of these five or six general principles to the various physical phenomena suffices to teach us what we may reasonably hope to know about them. The most remarkable example of this new mathematical physics is without doubt Maxwell's electro-magnetic theory of light. What is the ether? How are its molecules distributed? Do they attract or repel each other? Of these things we know nothing. But we know that this medium transmits both optical and electrical disturbances; we know that this transmission must take place in conformity with the general principles of mechanics and that suffices to establish the equations of the electro-magnetic field.

These principles are the boldly generalized results of experiment; but they appear to derive from their very generality a high degree of certainty. In fact, the greater the generality, the more frequent are the opportunities for verifying them, and such verifications, as they multiply, as they take the most varied and most unexpected forms, leave in the end no room for doubt.

Such is the second phase of the history of mathematical physics, and we have not yet left it. Shall we say that the first has been useless, that for fifty years science was on a wrong path and that there is nothing to do but to forget all that accumulation of effort which a vicious conception from the very beginning doomed to failure? By no means! Do you think the second period could have existed without the first? The hypothesis of central forces contained all the principles; it involved them as necessary consequences; it involved the principle of the conservation of energy, as well as that of mass, and the equality of action and reaction, and the law of least action, which appeared to be sure, not as experimental facts, but as theorems, and of which the statement had I know not how much greater precision and lesser generality than under their present form.

It is the mathematical physics of our fathers which has gradually made us familiar with these various principles, which has taught us to recognize them in the different garbs in which they are disguised. They have been compared with the results of experiment; it has been found necessary to change their expression in order to make them conform to the facts; thus they have been extended and strengthened. In this way they came to be regarded as experimental truths. The conception of central forces then became a useless support, or rather an encumbrance, inasmuch as it imposed upon the principles its own hypothetical character.

The bounds then are not broken, because they were elastic; but they have been extended. Our fathers who established them have not labored in vain; and in the science of to-day we recognize the general features of the outline they traced.

Are we now about to enter upon a third period? Are we on the eve of a second crisis? Are these principles on which we have reared everything about to fall in their turn? This has recently become a vital question.

Hearing me speak thus, you are thinking without doubt of radium, that great revolutionary of the present day; and indeed I shall return to it presently. But there is something else. It is not merely the conservation of energy that is concerned; all the other principles are in equal danger, as we shall see by successively passing them in review.

Let us begin with Carnot's principle. It is the only one which does not present itself as an immediate consequence of the hypothesis of central forces; quite to the contrary, indeed, it appears, if not actually to contradict this hypothesis, at least not to be reconcilable with it without some effort. If physical phenomena were due exclusively to the motion of atoms the mutual attractions of which depend only on the distance, it would seem that all these phenomena should be reversible; if all the initial velocities were reversed, these atoms, if still subject to the same forces, should traverse their trajectories in the opposite direction, just as the earth would describe backward this same elliptical orbit that it now describes forward, if the initial conditions of its motion had been reversed. Thus, if a physical phenomenon is possible, the inverse phenomenon should be equally possible, and one should be able to retrace the course of time. Now, it is not so in nature, and this it is precisely that the principle of Carnot teaches us; heat may pass from a hot body to a cold; it is impossible to compel it to take the opposite route and to re-establish differences of temperature which have disappeared. Motion can be entirely destroyed and transformed into heat by friction; the converse transformation can only occur partially.

Efforts have been made to reconcile this apparent contradiction. If the world tends toward uniformity, it is not because its ultimate parts, though diversified at the start, tend to become less and less different; it is because moving at random they become mixed. To an eye which could distinguish all the elements, the variety would remain always as great; every grain of this powder retains its originality and does not fashion itself after its neighbors; but as the mixture becomes more and more perfect, our rough senses perceive only uniformity. That is why, for example, temperatures tend to equalize themselves, without its being possible to go back.

A drop of wine, let us say, falls into a glass of water; whatever the internal motion of the liquid, we shall soon see it assume a uniformly roseate hue, and from then on no possible shaking of the vessel would seem to be capable of again separating the wine and the water. Here, then, we have what may be the type of the irreversible phenomenon of physics: to hide a grain of barley in a great mass of wheat would be easy; to find it again and to remove it is practically impossible. All this has been explained by Maxwell and Boltzmann, but the man who has put it most clearly was Gibbs, in a book too little read because it is a little difficult to read, in his Elements of Statistical Mechanics.

To those who take this point of view, Carnot's principle is an imperfect principle, a sort of concession to the frailty of our senses; it is because our eyes are too coarse that we do not distinguish the elements of the mixture; it is because our hands are too coarse that we can not compel them to separate; the imaginary demon of Maxwell, who can pick out the molecules one by one, would be quite able to constrain the world to move backwards. That it should return of its own accord is not impossible; it is only infinitely improbable; the chances are that we should wait a long time for that combination of circumstances which would permit a retrogression; but, sooner or later, they will occur, after years, the number of which would require millions of figures. These reservations, however, all remained theoretical; they caused little uneasiness and Carnot's principle preserved all of its practical value.

But now here is where the scene changes. The biologist, armed with his microscope, has for a long time noticed in his preparations certain irregular motions of small particles in suspension; this is known as Brown's motion. He believed at first that it was a phenomenon of life, but he soon saw that inanimate bodies hopped about with no less ardor than others; he then turned the matter over to the physicists. Unfortunately, the physicists did not become interested in the question for a long time.

Light is concentrated, so they argued, in order to illuminate the microscopical preparation; light involves heat, and this causes differences in temperature and these produce internal currents in the liquid, which bring about the motions referred to.

M. Gouy had the idea of looking a little more closely, and thought he saw that this explanation was untenable; that the motion becomes more active as the particles become smaller, but that they are uninfluenced by the manner of lighting. If, then, these motions do not cease, or, rather, if they come into existence incessantly, without borrowing from any external source of energy, what must we think? We must surely not abandon on this account the conservation of energy; but we see before our eyes motion transformed into heat by friction and conversely heat changing into motion, and all without any sort of loss, since the motion continues forever. It is the contradiction of Carnot's principle. If such is the case, we need no longer the infinitely keen eye of Maxwell's demon in order to see the world move backward; our microscope suffices. The larger bodies, those of a tenth of a millimeter, for example, are bombarded from all sides by the moving atoms, but they do not stir, because these shocks are so numerous that the law of probabilities requires them to compensate each other; but the smaller particles are hit too rarely to have this compensation take place with any degree of certainty and are thus incessantly tossed about. And so one of our principles is already in danger.

Let us consider the principle of relativity; this principle is not only confirmed by our daily experience, not only is it the necessary consequence of the hypothesis of central forces, but it appeals to our common sense with irresistible force. And yet it also is being fiercely attacked. Let us think of two electrified bodies; although they seem to be at rest, they are, both of them, carried along with the motion of the earth; Rowland has shown us that an electric charge in motion is equivalent to a current; these two charged bodies, then, are equivalent to two parallel currents in the same direction; these two currents should attract each other. By measuring this attraction we should be measuring the velocity of the earth; not its velocity relative to the sun and the stars, but its absolute velocity.

I know what will be said; it is not its absolute velocity; it is its velocity relative to the ether. But, how unsatisfactory that is! Is it not clear that with this interpretation, nothing could be inferred from the principle? It could no longer teach us anything, simply because it would no longer fear any contradiction. Whenever we have succeeded in measuring anything, we would always be free to say that it is not the absolute velocity, and if it is not the velocity relative to the ether, it might always be the velocity relative to some new unknown fluid with which we might fill all space.

And then experiment, too, has taken upon itself to refute this interpretation of the principle of relativity; all the attempts to measure the velocity of the earth relative to the ether have led to negative results. Herein experimental physics has been more faithful to the principle than mathematical physics; the theorists would have dispensed with it readily in order to harmonize the other general points of view; but experimentation has insisted on confirming it. Methods were diversified; finally Michelson carried.precision to its utmost limits; nothing came of it. It is precisely to overcome this stubborness that to-day mathematicians are forced to employ all their ingenuity.

Their task was not easy, and if Lorentz has succeeded, it is only by an accumulation of hypotheses.

The most ingenious idea is that of local time. Let us imagine two observers, who wish to regulate their watches by means of optical signals; they exchange signals, but as they know that the transmission of light is not instantaneous, they are careful to cross them. When station B sees the signal from station A, its timepiece should not mark the same hour as that of station A at the moment the signal was sent, but this hour increased by a constant representing the time of transmission. Let us suppose, for example, that station A sends its signal at the moment when its time-piece marks the hour zero, and that station B receives it when its time-piece marks the hour t. The watches will be set, if the time t' is the time of transmission, and in order to verify it, station B in turn sends a signal at the instant when its time-piece is at zero; station A must then see it when its time-piece is at t. Then the watches are regulated.

And, indeed, they mark the same hour at the same physical instant, but under one condition, namely, that the two stations are stationary. Otherwise, the time of transmission will not be the same in the two directions, since the station A, for example, goes to meet the disturbance emanating from B, whereas station B flees before the disturbance emanating from A. Watches regulated in this way, therefore, will not mark the true time; they will mark what might be called the local time, so that one will gain on the other. It matters little, since we have no means of perceiving it. All the phenomena which take place at A, for example, will be behind time, but all just the same amount, and the observer will not notice it since his watch is also behind time; thus, in accordance with the principle of relativity he will have no means of ascertaining whether he is at rest or in absolute motion. Unfortunately this is not sufficient; additional hypotheses are necessary. We must admit that the moving bodies undergo a uniform contraction in the direction of the motion. One of the diameters of the earth, for example, is shortened by 1 / 200000000 as a result of our planet's motion, whereas the other diameter preserves its normal length. Thus we find the last minute differences accounted for. Then there is still the hypothesis concerning the forces. Forces, whatever their origin, weight as well as elasticity, will be reduced in a certain ratio in a world endowed with a uniform translatory motion; or rather that would happen for the components at right angles to the direction of translation; the parallel components will not change. Let us then return to our example of the two electrified bodies; they repel each other; but at the same time, if everything is carried along in a uniform translation, they are equivalent to two parallel currents in the same direction, which attract each other.

This electrodynamic attraction is, then, subtracted from the electrostatic repulsion, and the resultant repulsion is weaker than if the two bodies had been at rest. But since we must, in order to measure this repulsion, balance it by another force, and since all these other forces are reduced in the same ratio, we observe nothing. Everything, then, appears to be in order. But have all doubts been dissipated? What would happen if we could communicate by signals other than those of light, the velocity of propagation of which differed from that of light? If, after having regulated our watches by the optical method, we wished to verify the result by means of these new signals, we should observe discrepances due to the common translatory motion of the two stations. And are such signals inconceivable, if we take the view of Laplace, that universal gravitation is transmitted with a velocity a million times as great as that of light?

Thus the principle of relativity has in recent times been valiantly defended; but the very vigor of the defense shows how serious was the attack.

And now let us speak of the principle of Newton, concerning the equality of action and reaction. This principle is intimately connected with the preceding and it would seem that the fall of one would involve the fall of the other. Nor must we be surprised to find here again the same difficulties.

The electrical phenomena, it is thought, are due to displacements of small charged particles called electrons which are immersed in the medium we call ether. The motions of these electrons produce disturbances in the surrounding ether; these disturbances are propagated in all directions with the velocity of light, and other electrons initially at rest are displaced when the disturbance reaches the portions of the ether in which they lie. The electrons, then, act one upon the other, but this action is not direct; it takes place by mediation of the ether. Under these conditions, is it possible to have equality between action and reaction, at least for an observer who takes account only of the motion of matter, that is of the electrons, and who ignores that of the ether which he is unable to see? Evidently not. Even if the compensation were exact, it could not be instantaneous. The disturbance is propagated with a finite velocity; it reaches the second electron, therefore, only after the first has long been reduced to rest. This second electron will, then, after an interval, be subjected to the action of the first, but will certainly not at that moment react upon it, since there is no longer anything in the neighborhood of this first electron that stirs.

The analysis of the facts will allow us to become more definite. Let us imagine, for example, a Hertzian oscillator such as those used in wireless telegraphy; it sends energy in all directions; but we may attach to it a parabolic mirror, as was done by Hertz with his smallest oscillators, so as to send all the energy produced in a single direction. What then will happen according to the theory? Why, the apparatus will recoil as though it were a cannon and the projected energy a ball, and that contradicts the principle of Newton, since our present projectile has no mass; it is not matter, it is energy. It is the same, moreover, in the case of a light-house having a reflector, since light is merely a disturbance in the electro-magnetic field. This light-house would recoil, as though the light it sends forth were a projectile. What is the force that must produce this recoil? It is what is known as the Maxwell-Bartholdi pressure; it is very small, and to put it in evidence caused much trouble, even with the most sensitive radiometers; but it is sufficient for our purpose that it exists.

If all the energy issuing from our oscillator strikes a receiver, the latter will act as though it had received a physical shock, which in a sense will represent the compensation of the oscillator's recoil; the reaction will be equal to the action, but they will not be simultaneous; the receiver will advance, but not at the instant when the oscillator recoils. If the energy is propagated indefinitely without meeting a receiver, the compensation will never take place.

Shall we say that the space which separates the oscillator from the receiver and which the disturbance must traverse in passing from one to the other, is not empty, but is filled not only with ether, but with air, or even in inter-planetary space with some subtile, yet ponderable fluid; that this matter receives the shock, as does the receiver, at the moment the energy reaches it, and recoils, when the disturbance leaves it? That would save Newton's principle, but it is not true. If the energy during its propagation remained always attached to some material substratum, this matter would carry the light along with it and Fizeau has shown, at least for the air, that there is nothing of the kind. Michelson and Morley have since confirmed this. We might also suppose that the motions of matter proper were exactly compensated by those of the ether; but that would lead us to the same considerations as those made a moment ago. The principle, if thus interpreted, could explain anything, since whatever the visible motions we could imagine hypothetical motions to compensate them. But if it can explain anything, it will allow us to foretell nothing; it will not allow us to choose between the various possible hypotheses, since it explains everything in advance. It therefore becomes useless.

And then the suppositions that must be made concerning the motions of the ether are not very satisfactory. If the electric charges were doubled, it would be natural to suppose that the velocities of the atoms of the ether also became twice as great, and for the compensation it would be necessary that the mean velocity of the ether become four times as great.

This is why I have for a long time thought that these consequences of the theory, which contradict Newton's principle, would some day be abandoned; and yet the recent experiments on the motion of the electrons emitted by radium seem rather to confirm them.

I now come to Lavoisier's principle concerning the conservation of mass. This is certainly a principle which cannot be tampered with without shaking the science of mechanics. And still there are persons who think that it seems true to us only because in mechanics we consider only moderate velocities, and that it would cease to be so for bodies having velocities comparable with that of light. Now, such velocities are at present believed to have been realized; the cathode rays and those of radium would seem to be formed of very minute particles or electrons that move with velocities that are no doubt less than that of light, but which appear to be about one tenth or one third of it.

These rays can be deflected either by an electric or by a magnetic field, and by comparing these deflections it is possible to measure both the velocity of the electrons and their mass (or rather the ratio of their mass to their charge). But it was found that as soon as these velocities approached that of light a correction was necessary. Since these particles are electrified, they cannot be displaced without disturbing the ether; to put them in motion, it is necessary to overcome a double inertia, that of the particle itself and that of the ether. The total or apparent mass that is measured is then composed of two parts: the real or mechanical mass of the particle and the electrodynamic mass representing the inertia of the ether.

Now, the calculations of Abraham and the experiments of Kaufmann have shown that the mechanical mass properly so called is nothing, and that the mass of the electrons, at least of the negative electrons, is purely of electrodynamic origin. This is what compels us to change our definition of mass; we can no longer distinguish between the mechanical mass and the electrodynamic mass, because then the first would have to vanish; there is no other mass than the electrodynamic inertia; but in this case, the mass can no longer be constant; it increases with the velocity; and indeed it depends on the direction, and a body having a considerable velocity will not oppose the same inertia to forces tending to turn it off its path that it opposes to those tending to accelerate or retard its motion.

There is indeed another resource: the ultimate elements of bodies are electrons, some with a negative charge, others with a positive charge. It is understood that the negative electrons have no mass; but the positive electrons, from what little is known of them, would seem to be much larger. They perhaps have besides their electrodynamic mass a true mechanical mass. The real mass of a body would then be the sum of the mechanical masses of its positive electrons, the negative electrons would not count; the mass defined in this way might still be constant.

Alas, this resource is also denied. Let us recall what we said concerning the principle of relativity and the efforts made to save it. And it is not only a principle that is to be saved; the indubitable results of Michelson's experiments are involved. And so, as was above seen, Lorentz, to account for these results, was obliged to suppose that all forces, whatever their origin, are reduced in the same ratio in a medium having a uniform translatory motion. But that is not sufficient; it is not enough that this should take place for the real forces, it must also be the same in the case of the forces of inertia; it is necessary, therefore — so he says — that *the masses of all particles be influenced by a translation in the same degree as the electromagnetic masses of the electrons.*

Hence, the mechanical masses must vary according to the same laws as the electrodynamic; they can then not be constant.

Do I need to remark that the fall of Lavoisier's principle carries with it that of Newton's? The latter implies that the center of gravity of an isolated system moves in a straight line; but if there no longer exists a constant mass, there no longer exists a center of gravity; indeed the phrase would be meaningless. This is why I said above that the experiments on cathode rays seemed to justify the doubts of Lorentz concerning Newton's principle.

From all these results, if they were to be confirmed, would issue a wholly new mechanics which would be characterized above all by this fact, that there could be no velocity greater than that of light,* any more than a temperature below that of absolute zero. For an observer, participating himself in a motion of translation of which he has no suspicion, no apparent velocity could surpass that of light, and this would be a contradiction, unless one recalls the fact that this observer does not use the same sort of timepiece as that used by a stationary observer, but rather a watch giving the "local time."

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* Because bodies would oppose an increasing inertia to the causes that would tend to accelerate their motion; and when approaching the velocity of light, this inertia would become infinite.

Here we are then face to face with a question, of which I shall confine myself to the mere statement. If there is no longer any mass what becomes of Newton's law?

Mass has two aspects: it is at the same time a coefficient of inertia and an attracting mass entering as a factor into Newton's law of attraction. If the coefficient of inertia is not constant, can the attracting mass be constant? This is the question.

The principle of the conservation of energy at least still remained and appeared more finely established. Shall I recall to your minds how it too was thrown into discredit? That event made more noise than the preceding; the journals are full of it. Ever since the first work of Becquerel, and above all after the Curies had discovered radium, it was seen that every radioactive substance was an inexhaustible source of radiation. Its activity seemed to continue without change through months and years. That is already a strain on the principles; these radiations in fact were energy, and from the same piece of radium came forth this energy and it came forth indefinitely. But these quantities of energy were too minute to be measured; at least that was the belief, and the matter caused little uneasiness.

The scene changed when Curie thought of placing the radium in a calorimeter. It was then seen that the quantity of heat continuously generated was very considerable.

The explanations advanced were numerous; but in a case of this kind it is not possible to say that an abundance of good does no harm: as long as one explanation has not displaced the others we can not be sure that any one of them is good. For some time, however, one of these explanations seems to be gaining the upper hand and we may reasonably hope that we hold the key to the mystery.

Sir W. Ramsey has attempted to show that radium is transformed, that it contains an enormous amount of energy, but not an inexhaustible amount. The transformation of radium must then produce a million times as much heat as any known transformation; the radium would be exhausted in 1250 years; that is not long, but you see that we are at least sure of being bound to the present state of affairs for some hundreds of years. While we wait our doubts subsist.

In the midst of such ruin, what remains standing? The principle of least action up to now is intact, and Larmor appears to think that it will long survive the others. It is in fact more vague and even more general.

In the presence of this general collapse of principles, what attitude should mathematical physics take? First of all, before becoming too excited, it is well to ask whether all this is really true. All this disparagement of principles is encountered only in the case of the infinitely small; the microscope is needed to see Brown's motion, the electrons are rather tiny, radium is very rare and never more than a few milligrams are together; and then we can ask whether by the side of the minute thing that was observed, there was not another minute thing which was not noticed and which counterbalanced the first.

The question is surely debatable, and apparently only experiment can solve it. We should merely have to turn the matter over to the experimenters and, while waiting for them definitely to settle the controversy, not to trouble ourselves with these disquieting problems, and to keep quietly at our work, as though the principles were still unchallenged. We certainly have enough to do without leaving the domain where they can be applied with all certainty; we have enough to keep us busy during this period of doubt.

And yet is it really true that we can do nothing to relieve science of these doubts? It must indeed be said that it has not been experimental physics alone that has brought them into existence; mathematical physics has contributed its share. It was the experimenters who saw radium emit energy; but the theorists were the ones who brought to light all the difficulties inherent in the propagation of light through a moving medium; had it not been for them, they probably would not have been noticed. They have, then, done their best to embarrass us; it is no more than just that they should help us to extricate ourselves.

They must subject to a searching criticism all the new conceptions that I have outlined to-day, nor must they abandon the principles except after a loyal effort to save them. What can they do in this direction? That is what I shall seek to explain.

Among the most interesting problems of mathematical physics a place should be set apart for those that belong to the kinetic theory of gases.

Much has already been done toward their solution, but much remains to do. This theory is an everlasting paradox. We have reversibility in the premises and irreversibility in the conclusions, and a deep chasm between the two. Will statistical considerations, and the law of large numbers, suffice to fill it up? Many points still remain obscure to which it will be necessary to return, and that without doubt several times. In clearing them up, the meaning of Carnot's principle will be better understood, and its general position in dynamics; and we shall be better able to interpret the curious experiment of Gouy to which I referred above.

Should we not also make an effort to obtain a more satisfactory theory of the electrodynamics of moving bodies? It is here, above all, as I indicated sufficiently a short time ago, that the difficulties accumulate; even though we heap up hypotheses, we can not satisfy all the principles at once; no one has succeeded so far in saving some without sacrificing others. But all hope of obtaining better results is not yet lost. Let us, then, take the theory of Lorentz. Let us turn it over and over, let us modify it little by little, and all will be well, perhaps.

Indeed, instead of supposing that bodies in motion undergo a contraction in the direction of motion and that this contraction is the same whatever the nature of these bodies and the forces to which they are subjected, could not a simpler and more natural hypothesis be made? One might suppose, for example, that it is the ether which changes when it is in relative motion with respect to the material substance which passes through it; that, when thus modified, it no longer transmits the disturbances with the same velocity in all directions. It would transmit more rapidly those disturbances which are being propagated parallel to the motion of the substance, be it in the same direction or in the opposite, and less rapidly those which are propagated at right angles. The wave surfaces would then no longer be spheres, but ellipsoids, and one could do without this extraordinary contraction of all bodies.

I am giving this only by way of example, since the modifications which could be tried are evidently susceptible of infinite variation.

It is possible also that astronomy may some day furnish us with data on this point: she it was, in fact, who raised the question by making known to us the phenomenon of the aberration of light. If the theory of the aberration of light is roughly constructed, a curious result is arrived at. The apparent positions of the stars differ from their real positions by reason of the earth's motion, and since this motion is variable the apparent positions vary. The real position we are unable to ascertain, but we can observe the variations of the apparent position. Observations on aberration then show us, not the earth's motion, but the variations of this motion. They can, therefore, teach us nothing concerning the absolute motion of the earth.

These at any rate are the facts under a first approximation; but such would no longer be the case if we could observe the thousandth part of a second. It would then be seen that the variation in the apparent motion of the star depends not only on the variation in the earth's motion, a variation which is well known, since it is the motion of our globe in its elliptical orbit, but also on the mean value of this motion, so that the constant of aberration would not be quite the same for all stars, and that the differences would make known to us the earth's absolute motion in space.

This would be, in another form, the end of the principle of relativity. We are far, it is true, from being able to observe the thousandth part of a second, but after all, say some, the earth's total absolute velocity is perhaps much greater than its velocity relative to the sun; if it were for instance 300 km. per second, instead of 30 km., that would suffice to make the phenomenon observable.

I believe that by reasoning in this manner we carry simplicity in the theory of aberration too far. Michelson has shown, as I have said, that the methods of physics are powerless to put absolute motion in evidence; I am convinced that in the case of astronomical methods it will be the same, no matter how far precision may be carried.

However that may be, the data which astronomy will furnish in this direction will one day be valuable to the physicist. In the meantime I believe that the theorists, keeping in mind the experiments of Michelson, may count on a negative result, and that they would do useful work by constructing a theory of aberration which takes account of it in advance.

But let us return to the earth. There, too, we can help the experimenters. We can, for example, prepare the way by studying thoroughly the dynamics of the electrons; not, be it well understood, by starting from a single hypothesis, but by multiplying the hypotheses as much as possible. It would then be the part of the physicist to use our work in searching for the crucial experiment which would decide between them.

This dynamics of the electrons can be approached from many sides; but among the roads that lead there, there is one which has been somewhat neglected, and yet it is one of those that promise us the most surprises. It is the motion of electrons that produces the lines of the spectrum; this is proved by the phenomenon of Zeemann; what vibrates in an incandescent body is affected by a magnet, and is hence electrified. This is a first very important point; but no one has gone into the question any further. Why are the lines of the spectrum distributed according to a regular law? These laws have been studied by the experimenters in the greatest detail; they are very precise and comparatively simple. A first study of these arrangements brings to mind the harmonics encountered in acoustics; but the difference is great. Not only are the numbers of vibrations not successive multiples of a single number; but we even find nothing analogous to the roots of those transcendental equations, to which we are led by so many problems of mathematical physics: that of the vibrations of an elastic body of any shape, of the Hertzian oscillations in a generator of any form, the problem of Fourier on the cooling of a solid body.

The laws are simpler, but they are of an entirely different kind. To mention only one of these differences, for harmonics of high order the number of vibrations tends toward a finite limit, instead of increasing indefinitely.

This has not yet been explained, and I believe that here is one of the most important of nature's secrets. Lindemann has made a praiseworthy attempt, but in my opinion without success. This attempt should be renewed. We shall thus penetrate, so to speak, into the intimacies of matter. And, from the particular point of view that we occupy to-day, when we shall know why the vibrations of incandescent bodies differ in this way from the vibrations of ordinary elastic bodies, why the electrons do not behave like the matter with which we are familiar, we shall better understand the dynamics of the electrons and it will perhaps be easier for us to reconcile them with the principles.

Now suppose that all these efforts should fail (and when all is said, I do not believe they will), what should be done? Should we seek to rebuild these shattered principles by one stroke, as it were? That, evidently, is always possible, and I take back nothing of what I once said. "Did you not write," you might say, if you were seeking a quarrel with me, "did you not write that the principles, though they are of experimental origin, are now beyond the possibility of experimental attack, because they have become conventions? And now you come to tell us that the triumphs of the most recent experiments put these principles in danger."

Very well, I was right formerly, and I am not wrong to-day. I was right formerly, and what is taking place at present is another proof of it. Let us take, for example, the calorimetric experiment of Curie with radium. Is it possible to reconcile it with the principle of the conservation of energy? This has been attempted in many ways; but there is one among them to which I wish to call your attention; it is not the explanation which is tending to-day to prevail, but it is one of those that have been suggested. Radium is assumed to be only an intermediary, merely to store radiations of an unknown nature that fly through space in all directions, traversing all bodies except radium without being changed by this passage, and without exerting on them any action whatever. Radium alone can appropriate a little of their energy and then return it to us in various forms.

How useful this explanation is and how convenient! In the first place it is non-verifiable and hence irrefutable. Then, it can serve to account for any contradiction to Mayer's principle; it answers in advance not only the objection of Curie, but all other objections that the experimenters of the future may accumulate. This new and unknown energy could be used for anything.

That is exactly what I said, and by such means it is easy to show that our principle is safe from experimental attacks.

But then, what have we gained by this stroke? The principle is intact, but henceforth what is it good for? It enabled us to foresee that under such and such conditions we could count on a certain amount of energy; it imposed a limit; but now that there has been placed at our disposal this indefinite supply of new energy, we are no longer limited by anything; and, as I have also written, if a principle ceases to be productive, experiment, without contradicting it directly, would nevertheless condemn it.

That, then, is not what should be done. We should have to rebuild from the beginning. If we were driven to this necessity, we could easily console ourselves. We should not be obliged to conclude that science can never do aught but the work of a Penelope, that it can only raise ephemeral structures which it is soon forced to demolish completely with its own hands.

As I have said, we have already passed through a similar crisis. I have shown you that in the second mathematical physics, that of general principles, one finds traces of the first, that of central forces; it will be the same if we are to know a third. Just so with the animal that casts its outer shell, that bursts the skin that has become too small and grows a new one; under the new covering can always be recognized the essential traits of the organism that survives.

In what direction we are going to expand we are unable to foresee. Perhaps it is the kinetic theory of gases that will forge ahead and serve as a model for the others. In that case, the facts that appeared simple to us at first will be nothing more than the resultants of a very large number of elementary facts which the laws of probability alone would induce to work toward the same end. A physical law would then assume an entirely new aspect; it would no longer be merely a differential equation, it would assume the character of a statistical law.

Perhaps too we shall have to construct an entirely new mechanics, which we can only just get a glimpse of, where, the inertia increasing with the velocity, the velocity of light would be a limit beyond which it would be impossible to go. The ordinary, simpler mechanics would remain a first approximation since it would be valid for velocities that are not too great, so that the old dynamics would be found in the new. We should have no reason to regret that we believed in the older principles, and indeed since the velocities that are too great for the old formulas will always be exceptional, the safest thing to do in practice would be to act as though we continued to believe in them. They are so useful that a place should be saved for them. To wish to banish them altogether would be to deprive oneself of a valuable weapon. I hasten to say, in closing, that we are not yet at that pass, and that nothing proves as yet that they will not come out of the fray victorious and intact.

SHORTER NOTICES.

Carl Gustav Jacob Jacobi. Festschrift zur Feier der hundertsten Wiederkehr seines Geburtstages. Mit einem Bildnis und dem Faksimile eines Briefes. By LEO KÖNIGSBERGER. Leipzig, B. G. Teubner, 1904. 8vo. xvii + 554 pp.

IT is a pleasure as well as a duty for posterity to honor and keep green the memory of those great men of science whose discoveries have contributed to human happiness or whose profound meditations have opened up new and fruitful lines of research. To the latter class belongs Jacobi, the hundredth anniversity of whose birthday fell on December 10, 1904. The third International congress of mathematicians, held that year, determined to celebrate this event with due solemnity and Professor Königsberger was asked to prepare a commemorative address. No better choice of speaker could be made; for Professor Königsberger besides having published a noteworthy history of the elliptic functions for the years 1826–29 has contributed numerous important papers on the abelian transcendants, on differential equations and on principles of mechanics, all subjects which occupied Jacobi's unceasing attention.

The exceptional position that Jacobi occupies in the history of German mathematics makes the story of his life doubly interesting. At the time when he began his career the condition of mathematics in Germany was truly deplorable. The whole mathematical genius of the Teutonic people seems to have been absorbed by a single individual, Gauss, leaving a mere pittance to be thinly distributed among his contemporaries. Moreover the cold and reserved nature of Gauss, his uncommunicative if not secretive disposition, his unvarying practice of publishing nothing that was not perfect in form and complete in content, all contributed most unhappily in reducing to a minumum the mighty influence he should have exerted on his countrymen during the first quarter of the nineteenth century.

With the advent of Jacobi a new era began; a new generaation of mathematicians sprang up with a suddenness and brilliancy which is astounding. Jacobi was presently joined by Steiner, Dirichlet, Plücker and others, and Germany began the triumphal march which soon led her to the first rank in the world of mathematics.

The most prominent figure in this new generation is Jacobi.

His epoch-making discoveries in the theory of elliptic functions when only twenty-three years of age excited the admiration of all Europe, while his many and profound researches rapidly rendered him one of the most celebrated mathematicians of his day. But Germany had had great mathematicians before; Leibniz, Euler, the Bernoullis, not to mention later ones. What distinguishes Jacobi from his predecessors is the fact that he was a great teacher. In this respect he was the very opposite of his great contemporary Gauss, who disliked to teach, and who was anything but inspiring. Jacobi's lectures were models of clearness; the enthusiasm of the teacher kindled a like enthusiasm in his pupils. He took the unprecedented step of making his more gifted students acquainted with his own unfinished investigations and stimulated them to attack problems suggested by these researches. The results were most gratifying, for ere long we see his pupils, Hesse, Goepel, Rosenhain, Richelot, and others, publishing important memoirs.

It is not our intention, however, to write a sketch of Jacobi's life, but rather to call attention to the interest and the lessons which it has for us. We feel sure that no reader will regret perusing Professor Königsberger's biography, which is obviously a labor of love and the result of long and patient research. A feature of the work deserves especial commendation, and is most unusual. All of Jacobi's principal papers are carefully analyzed and the growth and filiation of his mathematical ideas stand forth in bold relief. It has no doubt cost his biographer months of patient study to do this; but the results here given will prove of utmost service to future generations of mathematicians who seek a rapid orientation of the leading facts in any part of Jacobi's multitudinous and varied publications.

JAMES PIERPONT.

Uebungsbuch zum Studium der höheren Analysis. VON OSKAR SCHLÖMILCH. Erster Theil: *Aufgaben aus der Differentialrechnung.* Fifth edition, prepared by Dr. E. NAETSCH. Leipzig, B. G. Teubner, 1904. 8vo. viii + 372 pp.

THIS work whose first edition appeared more than thirty years ago has enjoyed a well-merited popularity, both on account of the variety and careful choice of its problems and also the elegance and instructiveness often exhibited in their solutions. Numerous additions have been made in this, the fifth edition, giving fresh interest to the work. The problems relating to curves and surfaces, maxima and minima, limits and infinite series, are especially praiseworthy.

JAMES PIERPONT.

A Course in Mathematical Analysis, by EDOUARD GOURSAT, Professor of Mathematics in the University of Paris; translated by EARLE RAYMOND HEDRICK, Professor of Mathematics in the University of Missouri. Volume I. Ginn and Company, Boston, 1905. 8vo. viii + 548 pp.

THE French edition of this work was published in 1902, and it was reviewed in the BULLETIN.* While it is true that advanced students of mathematics recognize the necessity of learning to read mathematical French and German, and equip themselves duly in this respect, the undergraduate finds the additional difficulty of a foreign language a serious handicap in the use of a mathematical text-book. And yet it is precisely for the undergraduate, the student in the second course in calculus and the first course in the theory of functions, who is perhaps preparing to specialize in applied mathematics and will not carry his study of analysis beyond the undergraduate courses, that Professor Goursat's book contains so much which is important but at present is not to be found in English text-books. Professor Hedrick has prepared the translation with great care and has made it a worthy reproduction of this standard work. In his preface he says: "Few alterations have been made from the French text. Slight changes of notation have been introduced occasionally for convenience, and several changes and additions have been made at the suggestion of Professor Goursat, who has very kindly interested himself in the work of translation." To the publishers is due much credit for the excellent typography of the book. Niceties of spacing and arrangement of the formulas, which hitherto have usually been neglected by American and English printers, here contribute to make the page extremely attractive.

WM. F. OSGOOD.

Elementary Modern Geometry. Part I.: *Experimental and Theoretical, Triangles and Parallels.* By H. G. WILLIS. Oxford, Clarendon Press, 1905. v + 236 pp.

THE order of sequence of the adjectives in the above title might lead to a misunderstanding; the subjects treated are those of a beginner's school geometry, but the method of treatment is rather unusual. Indeed, so many text-books on geometry are being published that the appearance of still another seems hardly justified, unless it presents something new. The first seventy pages are empirical and experimental to develop space intuition without any attempt at formal demonstration. Paper folding plays an important part and most of the constructions of ordinary geometry are actually made. Free use is made of ruler, compasses and superposition.

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* Cf. BULLETIN, ser. 2, vol. 9 (1902–03), p. 547.

After this somewhat extensive introduction, formal definitions, postulates and axioms are given, the latter including an axiom of continuity, an axiom of displacement and an axiom of rotation. The last axiom replaces Euclid's fifth postulate and is stated as follows: "If a straight line, after turning in a plane about various points in itself, coincides with its initial position, the algebraic sum of the angles turned through is zero or one or more straight angles." The definitions of a straight line and of a plane are those of Leibniz.

Demonstrations based on unexplained constructions are frequent in the book, even the possibility of the construction not being expressly postulated. The third chapter considers all these constructions and adds a good discussion of graphs, including the circle, hyperbola, sine curve (without trigonometry) and a long list of others as exercises, a good sample being

$$y^2 = x(x-2)^2(x+2).$$

The fourth chapter, of fifty pages, deals with parallels. Limiting values are discussed, and parallel lines are said to meet at infinity. The propositions are carefully worded, and the various theorems are shown to be all true or all false together. An alternate treatment, based on Playfair's statement of the fifth postulate is also given, but the two procedures should be more sharply separated, especially in the deductions drawn from the sum of the angles of a triangle. The subsequent propositions of the chapter are all discussed by both methods, but the proper selection of the demonstration which belongs to each is pointed out only in the preface.

A particularly instructive feature of the book is the long list of well-chosen examples and original questions, about two thousand being included. Among them are thirty-five on non-euclidean space, which are certainly novel, and of considerable educational value provided the pupil has not become so saturated with intuitional ideas derived from the first part that he cannot rid himself of them. It seems to me that in the hands of a skillful teacher the book would prove of value, although the only test is that of actual use. A school text in geometry which succeeds in making a direct and natural transition from the traditional programme to such books as von Staudt's and as Enriques's will be a welcome addition to our mathematical literature.

VIRGIL SNYDER.

Zwölf Vorlesungen über die Natur des Lichtes. Von J. CLASSEN. Leipzig, G. J. Göschen, 1905. x + 249 pp.

The author has already published through Göschen in the Sammlung Schubert three volumes on electricity, magnetism, and light which appeal to a technical public of mathematicians and physicists. The present volume, which is not in the Sammlung Schubert, is of a wholly different character. It is the reprint of a series of lectures given before a lay public and consequently freed of all technicalities whether mathematical or physical. The aim of the book is to set forth and discuss those simple experiments which begin with showing the rectilinear propagation of light and advance systematically and logically to the end of demonstrating the electromagnetic nature of light. A large range of phenomena is treated, including even the latest researches, such as those of Rubens, which put the electromagnetic theory upon a seemingly sure footing. A mathematical treatment of the subject built up in the same straightforward and logical manner would be highly useful, and better worth our attention here.

E. B. WILSON.

NOTES.

THE ninth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will be held at Stanford University on Saturday, February 24. Abstracts of papers intended for presentation should be in the hands of the Secretary of the Section, Professor G. A. Miller, Stanford University, as early as February 10.

AT the recent meeting of the Chicago Section, Professor T. F. HOLGATE, who has served as Secretary since the organization of the Section in 1897, withdrew from that office, which was thereupon filled by the election of Professor H. E. SLAUGHT. The next meeting of the Section falls on Saturday, April 14.

THE opening (January) number of volume 7 of the *Transactions of the American Mathematical Society* contains the following papers: "On the relation between the three parameter groups of a cubic space curve and a quadric surface," by A. B. COBLE; "On certain hyperabelian functions which are expressible by theta series," by J. I. HUTCHINSON; "On the form of a plane quintic curve with five cusps," by P. FIELD; "The symbolic treatment of differential geometry," by A. W. SMITH; "Groups whose orders are powers of a prime," by W. B. FITE; "Differential parameters of the first order," by H. MASCHKE; "The Kronecker-Gaussian curvature of hyperspace," by H. MASCHKE; "Groups containing only three operators which are squares," by G. A. MILLER; "Theorems converse to Riemann's on differential equations," by D. R. CURTISS; "General mean value and remainder theorems," by G. D. BIRKHOFF; "Determination of the abstract groups of order p^2qr; p, q, r being distinct primes," by O. E. GLENN; "Note on the differential invariants of a surface and of space," by C. N. HASKINS; "On improper multiple integrals," by JAMES PIERPONT.

THE opening (January) number of volume 28 of the *American Journal of Mathematics* contains: "On the quaternary linear homogeneous groups modulo p of order a multiple of p," by L. E. DICKSON; "On the integration of a system of differential equations in kinematics," by J. EIESLAND; "On the determination of the properties of the nodal curve of a unicursal ruled surface," by C. H. SISAM; "Certain surfaces with plane or spherical lines of curvature," by L. P. EISENHART; "The motion of a solid in an infinite liquid," by A. G. GRANVILLE.

THE January number (volume 7, number 2) of the *Annals of Mathematics* contains: "The harmonic analysis of the semicircle and of the ellipse," by A. E. KENNELLY; "Note on the possible number of operators of order 2 in a group of order 2^m," by G. A. MILLER; "A geometrical problem connected with the continuation of a power series," by H. MASCHKE; "On the determination of a catenary with given directrix and passing through two given points," by H. F. MacNeish; "Concerning the discontinuous solution in the problem of the minimum surface of revolution," by H. F. MacNeish; "Introduction to the theory of Fourier's series" (to be continued), by M. Bôcher.

At the meeting of the London mathematical society held on December 14 the following papers were read: "On well-ordered aggregates," by Professor C. A. Dixon; "Tables of coefficients for Lagrange's interpolation formula," by Col. R. L. Hippisley: "On the representation of certain asymptotic series as convergent continued fractions," by Professor L. J. Rogers; "On a new cubic connected with the triangle," by Mr. H. L. Trachtenburg; "Some difficulties in the theory of transfinite numbers and order types," by Hon. B. A. W. Russell; "The imaginary in geometry," by Mr. J. L. S. Hatton.

The third annual meeting of the association of Ohio teachers of mathematics and science was held at the Ohio State University, Columbus, December 28, 1905. Two sessions were held in each section. The following papers in mathematics were read: "Do the mathematical courses in literary colleges properly fit for the mathematics of engineering?" by C. Hornung; "Recent contributions marking a real advance in the theory of mathematics teaching, by Miss A. H. Palmié; "Sir Isaac Newton, an estimate," by C. L. Arnold; "A contribution from non-euclidean geometry to school spherics," by G. B. Halsted; "Symbolism in mathematics," by Miss H. E. Glazier; "The character of mathematics teaching in the high school," by W. T. Heilman; "The influence of college entrance certificates on the teaching of mathematics in the high schools," by H. E. Giles; "Report of committee on syllabus," by G. B. Halsted. The officers for the present year are: president, W. H. Wilson; vice-president, J. O. Williams; secretary, T. E. McKinney.

At the last meeting of the central association of science and mathematics teachers, the following officers were elected: president, O. W. Caldwell; secretary, C. M. Turton; treasurer, C. W. D. Parsons. The officers of the mathematics section are: chairman, J. V. Collins; vice-chairman, H. E. Slaught; secretary, Miss Mabel Sykes.

THE mathematical section of the California teachers association held two meetings at Berkeley on December 26 and 28. The following papers were read: "How to cultivate the power to think mathematically," by I. STRINGHAM; "Vital questions for teachers of secondary mathematics," by J. B. CLARKE. During the business meeting the section adopted *School Science and Mathematics* as its official journal, and re-elected the following officers: president, G. A. MILLER; vice president, W. H. BAKER; secretary, J. F. SMITH.

AT the meeting of the mathematics section of the Indiana state teachers association held at Indianapolis December 27, a committee of five with Professor D. A. ROTHROCK, of Indiana University, as chairman was appointed to perfect a plan of organization of the mathematics teachers of the state into an Indiana association of mathematics teachers. The proposed association will meet for organization at Indianapolis, March 30.

THE *Annals of Mathematics*, in pursuance of the policy announced in 1899, when the responsibility for the journal was assumed by Harvard University, has published a considerable number of articles whose purpose is not so much to present new results as to render more accessible, or put in better form, some important portion of mathematical theory. It may not be generally known that reprints of most of these expository articles can be obtained from the Publication Office of Harvard University, 2 University Hall, Cambridge, Mass., at prices ranging from thirty-five to seventy-five cents. The list of titles is as follows:

Galois's theory of algebraic equations. By JAMES PIERPONT. 67 pages.

The theory of linear dependence. By MAXIME BÔCHER. 16 pages.

Sufficient conditions in the calculus of variations. By W. F. OSGOOD. 25 pages.

Space of constant curvature. By F. S. WOODS. 42 pages.

The integral as the limit of a sum, and a theorem of Duhamel's. By W. F. OSGOOD. 18 pages.

The continuum as a type of order: an exposition of the modern theory. With an appendix on the transfinite numbers. By E. V. HUNTINGTON. 63 pages.

Introduction to the theory of Fourier's series. By MAXIME BÔCHER. (In press.)

The following pamphlets are also on sale at the same place:

Regular points of linear differential equations of the second order. By MAXIME BÔCHER. 23 pages.

Introduction to infinite series. By W. F. OSGOOD. 71 pages.

THE following books are announced by the firm of B. G. Teubner, Leipzig, to be in press, and will probably appear in a few weeks: Encyklopädie der mathematischen Wissenschaften, II, 1; III, 1, 2; IV, 2; V, 1; VI, 1, 2; Encyclopédie des sciences mathématiques, tome I, volumes 1, 2, 3, 4; H. POINCARÉ, Der Wert der Wissenschaft, translated by E. WEBER; Serret-Harnack, Lehrbuch der Differential- und Integralrechnung, in 3 Bänden, dritte Auflage, by G. SCHEFFERS. The German translation of Enriques's Questioni riguardanti la geometria elementare, by H. FLEISCHER, will be enlarged to two volumes, of which the second, Die geometrischen Aufgaben, is in the press. The first volume is promised for next fall.

AT the annual public meeting of the Paris academy of sciences, held on December 18, 1905, the following prizes were awarded for mathematical memoirs: the Francoeur prize (fr. 1000) to H. STOUFF; the Poncelet prize (fr. 2000) to N. LALLEMAND; two Petit d'Ormay prizes (each fr. 10000), one to É. BOREL, the other to M. COSSANTIN. The subjects for which these prizes were offered were announced in the BULLETIN, volume 11, page 327.

UNIVERSITY OF PARIS. (The following courses are given during the present semester.)—By Professor G. DARBOUX: General principles of differential geometry, curvilinear coördinates and quadratic differential forms, two hours.—By Professor E. GOURSAT: Elements of the theory of analytic functions, two hours.—By Professor P. PAINLEVÉ: General laws of motion and of equilibrium, two hours.—By Professor P. APPELL: Elements of mathematics preparatory to mechanics and physics, two hours.—By Professor L. RAFFY: Geometric applications of analysis, two hours.—By Professor H. POINCARÉ: Planetary perturbations and development of the perturbation function, two hours.—By Professor J. BOUSSINESQ: Mechanical properties of fluids, two hours.—By Professor E. BOREL: Selected chapters in the theory of integral functions, one hour.—By Professor G. Koenigs: General kinematics, two hours.—By Dr. P. Puiseux: Stars and eclipses, one hour. Mathematical conferences will be held by Professors Goursat, Hadamard, Raffy, Drs. Blutel and Servant.

Oxford University (*Hilary term*).—By Professor W. Essen: Comparison of analytic and synthetic methods in the geometry of conics, two hours; Synthetic geometry of cubics, one hour.—By Professor E. B. Elliott: Elements of elliptic functions, two hours; Theory of numbers, II, one hour.—By Professor H. H. Turner: Elementary mathematical astronomy, two hours.—By Professor A. E. H. Love: Theory of potential, two hours; Elements of the calculus, two hours.—By Mr. J. W. Russell: Algebra of quantics, two hours.—By Mr. P. J. Kirkby: Higher algebra, one hour.—By Mr. A. L. Dixon: Calculus of finite differences, one hour.—By Mr. J. E. Campbell: Differential geometry, two hours.—By Mr. C. H. Sampson: Solid geomety, II, two hours.—By Mr. A. L. Pedder: Trigonometry, one hour.—By Mr. C. Leudesdorf: Geometry, two hours.—By Mr. A. E. Jolliffe: Analytic geometry, II, two hours.—By Mr. R. F. McNeile: Integral calculus, two hours.

University of Innsbruck. (The following courses are announced for the summer semester, 1906.)—By Professor K. Zindler: Analytic geometry of two and three dimensions, II, four hours; Selected chapters of line geometry, three hours.—By Professor J. Menger: Descriptive geometry with exercises, II, four hours. The successor of the late Professor O. Stolz has not yet been appointed.

Dr. H. Hahn, of the technical school at Vienna, has charge of the courses which were announced for the late Professor Otto Stolz for the present semester at the University of Innsbruck.

Dr. F. Cohn has been promoted to an associate professorship of mathematics and astronomy at the University of Königsberg.

Dr. H. von Koch, of the University of Stockholm, has been appointed professor of pure mathematics at the technical school of Stockholm.

Professor R. Müller, of the technical school of Berlin, has been granted a leave of absence for the present academic year.

Dr. G. Faber has been appointed docent in mathematics at the technical school at Karlsruhe.

Dr. E. Stübler has been appointed docent in mathematics at the technical school of Berlin.

Professor A. Pringsheim, of the University of Munich, and Professor R. Fricke, of the technical school at Brunswick, have been elected corresponding members of the Göttingen academy of sciences.

Professor K. Boehm, of the University of Heidelberg, and Professor J. Hurwitz, of the technical school at Zürich, have been granted leave of absence for the present semester on account of ill health.

Professor G. Bauer, of the University of Munich, celebrated his eighty-fifth birthday November 18.

Mr. Richard Morris, instructor in mathematics at Rutgers College, has been promoted to an associate professorship of graphics and applied mathematics.

Professor C. J. Joly, of the University of Dublin, and royal astronomer of Ireland, died in December, 1905, at the age of 42 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Achsel (R.). Ueber den Zahlbegriff bei Leibniz. Wilmersdorf, 1905. 4to. 20 pp. M. 1.50

Arndt (K.). Grundbegriffe der höheren Mathematik für Chemiker. Berlin, Mayer, 1905. 8vo. 60 pp. M. 1.50

Bortolotti (E.). Lezioni sul calcolo degli infinitesimi date nella R. Università di Modena. Modena, 1905. 8vo. 7 + 61 pp. L. 3.00

Ertelt (G.). Synthetische Beweise einiger Sätze aus der Theorie der Flächen zweiten Grades. (Progr.) Bielitz, 1905. 8vo. 21 pp.

Fuller (J.). New and original treatise on the geometrical development of round and oval cones. New York, 1905. 8vo. 53 pp. Cloth. $1.25

Greul (A.). Ueber Scharen von ∞^{2n} Kurven im R_{n+1}. Erster Teil: Der Fall $n=2$. (Diss., Greifswald.) Dresden, Teubner, 1905. 8vo. 32 pp.

Hepke (B.). Ueber kürzeste Transversalen zwischen Erzeugenden einer hyperboloidischen Regelschaar. (Diss.) Breslau, Fleischmann, 1905. 8vo. 78 pp. M. 2.00

Horn (J.). Gewöhnliche Differentialgleichungen beliebiger Ordnung. Leipzig, Göschen, 1905. 12mo. Cloth. (Sammlung Schubert L.) M. 10.00

Koppisch (A.). Zur Invariantentheorie der gewöhnlichen Differentialgleichung zweiter Ordnung. (Diss.) Greifswald, 1905. 8vo. 34 pp.

Lampe (E.). Rede zur Gedächtnisfeier für Guido Hauck am 17. Mai 1905 in der Halle der Technischen Hochschule zu Charlottenburg gehalten. Nebst der Rede am Sarge in der Halle des Friedhofs der Zwölfapostelgemeinde gehalten am 28. Januar 1905 von A. Parisius. Mit einem Bild von G. Hauck als Titelbild. Leipzig, Teubner, 1905. M. 0.80

Lobatchewsky (N. I.). Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèles. Nouvelle édition conforme à la première (1835). Paris, Hermann, 1905. 8vo. Fr. 5.00

Lodge (A.). Integral calculus for beginners. London, Bell, 1905. 16mo. 203 pp. Cloth. 4s. 6d.

Miller (J. R.). Significance of the mathematical elements in the philosophy of Plato. Chicago, 1905. 8vo. 96 pp. $1.00

Möllers (B.). Ueber Normalsysteme, die mit der Rotations- und Schraubenfläche der Traktrix zusammenhängen. Münster, 1905. 8vo. 85 pp. M. 2.50

Müller (E.). See Schröder (E.).

Parisius (A.). See Lampe (E.).

Petri (K.). Ueber die in der Theorie der ternären kubischen Formen auftretenden Konnexe. (Diss.). München, 1903. 8vo. 44 pp.

Pirondini (G.). Osservazioni relative all'integrazione delle equazioni differenziali delle varie specie. Memoria. Parma, Rossi-Ubaldi, 1905. 8vo. 44 pp.

Remoundos (G.). Sur les zéros d'une classe de fonctions transcendantes. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 80 pp.

Schröder (E.). Vorlesungen über die Algebra der Logik (exakte Logik). (In 3 Bänden.) Vol. II: Zweite (Schluss-) Lieferung. Herausgegeben im Auftrage der Deutschen Mathematiker-Vereinigung von E. Müller. Mit einem Bildnisse des Verfassers. Leipzig, Teubner, 1905. 8vo. 32 + 206 pp.

Schubert (H.). Die Ganzzahligkeit in der algebraischen Geometrie. Den Teilnehmern der 48. Versammlung Deutscher Philologen und Schulmänner zu Hamburg 1905 als Festgabe dargeboten von dem Ortskomitee. Leipzig, Spamer, 1905.

SCHUH (F.). Vergelijkend overzicht der methoden ter bepaling van aantallen vlakke krommen. Amsterdam, 1905. 8vo. 11 + 218 pp. M. 6.00

STAUDE (O.). Analytische Geometrie des Punktes, der geraden Linie und der Ebene. Ein Handbuch zum Gebrauche neben den akademischen Vorlesungen. Leipzig, Teubner, 1905. 8vo. 8 + 448 pp. Cloth. (B. G. Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band XVI.) M. 14.00

SÜSS (A.). Die Gruppen, die mit der allgemeinen projektiven Gruppe der Ebene gleiche Zusammensetzung haben. (Diss., Greifswald.) Dresden, Teubner, 1905. 8vo. 34 pp.

TANNERY (J.). Leçons d'algèbre et d'analyse, à l'usage des élèves de la classe de mathématiques spéciales. (En 2 volumes.) Vol. I. Paris, Gauthier-Villars, 1906. 8vo. 7 + 423 pp. Fr. 12.00

WIELEITNER (H.). Theorie der ebenen algebraischen Kurven höherer Ordnung. Leipzig, Göschen, 1905. 12mo. Cloth. (Sammlung Schubert XLIII.) M. 10.00

WOLFF (G.). Ueber Gruppen der Reste eines beliebigen Moduls im algebraischen Zahlkörper. (Diss.). Göttingen, 1905. 8vo. 46 pp. M. 1.50

II. ELEMENTARY MATHEMATICS.

BAUR (L.). Methodisch geordnete Aufgaben aus dem Gebiete der Planimetrie und Stereometrie mit Angabe und Begründung der Berechnungsformeln für Schul- und Selbstunterricht. Stuttgart, Steinkopf, 1905. 8vo. 6 + 167 pp. M. 2.40

BORCHARDT (G. W.). Key to a new trigonometry for schools. Part 2. London, Bell, 1905. 8vo. 244 pp. 5s.

BORTOLOTTI (E.). Aritmetica generale ed algebra per la 1^a classe liceale. Roma, Segati, 1905. 8vo. 120 pp. L. 1.50

BOURLET (C.). Précis d'algèbre, contenant 573 exercices et problèmes, rédigé conformément aux programmes du 31 mai 1902 et du 27 juillet 1905. Paris, Hachette, 1905. 16mo. 445 pp. Fr. 2.50

CABREIRA (A.). Note sur les rapports des solides. Coimbra, 1905. 8vo. 11 pp.

CONANT (L. L.). Original exercises in plane and solid geometry. New York, American Book Company, [1905]. 12mo. 124 pp. Cloth. $0.50

DASSEN (C. C.). Tratado elemental de algebra de acuerdo con el concepto moderno de esta ciencia y los métodos mas rigurosos. Buenos Ayres, Hermanos, 1905. 12mo. 18 + 528 pp.

——. Tratado elemental de geometría euclídea de acuerdo con las ideas modernas y métodos mas rigurosos. Buenos Ayres, Hermanos, 1905. 12mo. 15 + 470 pp.

DAVISON (C.). Elements of solid geometry. Cambridge University Press, 1905. 8vo. 130 pp. 2s. 6d.

Fechner (H.). Aufgaben für den Unterricht in der Buchstabenrechnung (Algebra). 5te vermehrte Auflage. Berlin, Schultze, 1905. 8vo. 4 + 266 pp. M. 2.50

Grévy (A.). Algèbre, à l'usage des élèves des classes de quatrième B à première C et D. 4e édition. Paris, Vuibert et Nony, 1905. 16mo. 379 pp.

——. Géométrie dans l'espace, conforme au programme du 27 juillet 1905, à l'usage des élèves du deuxième cycle A et B. Paris, Vuibert et Nony, 1905. 16mo. Pp. 281–396.

——. Traité de géométrie, à l'usage des élèves de mathématiques A et B (programme du 31 mai 1902) et des candidats aux Ecoles de Saint-Cyr et navale. Paris, Vuibert et Nony, 1905. 8vo. 8 + 598 pp. Fr. 6.00

Gubler (E.). Aufgaben aus der allgemeinen Arithmetik und Algebra für Mittelschulen. 3tes Heft. Zürich, 1905. 8vo. 48 pp. M. 0.80

Hawkes (H. E.). Advanced algebra. (Mathematical texts.) Boston, Ginn, 1905. 8vo. 13 + 285 pp. Cloth. $1.50

Hecht (C.). Ueber Notwendigkeit und Gestaltung des Unterrichts in der elementaren Mathematik an der höheren Mädchenschule. Vortrag. Bielefeld, Velhagen, 1905. 8vo. 24 pp. M. 0.50

Kammer (O.). Inversionen bei Permutationen mit Wiederholung. (Progr.) Alsfeld, 1905. 4to. 21 pp.

Kosmik (K.). Das Dreieck, vornehmlich betrachtet in Bezug auf die ihm zukommenden besonderen Kreise und deren Radien. (Progr.) Mähr.-Weisskirchen, 1905. 8vo. 27 pp.

L. (G.). Repertorio di matematiche e fisica elementari. 2a edizione. Livorno, Giusti, 1906. 16mo. 156 pp. L. 1.00

Limarzi (N.). Compendio pratico di aritmetica, algebra e geometria ad uso delle scuole tecniche e commerciali. Torino, Bertero, 1905. 8vo. 64 pp.

Lodge (O.). Easy mathematics, chiefly arithmetic; being a collection of hints to teachers, parents, self-taught students and adults, and containing a summary or indication of most things in elementary mathematics useful to be known. London and New York, Macmillan, 1905. 12mo. 15 + 436 pp. Cloth. $1.10

Lyman (E. A.). Advanced arithmetic. For secondary schools and normal schools. New York, American Book Company [1905]. $0.75

Martini Zuccagni (A.). Trattato di algebra complementare ad uso degli istituti tecnici, con molti esercizi risoluti. Livorno, Giusti, 1905. 8vo. 8 + 283 pp. L. 3.00

Mathematical questions and solutions from "Educational Times." New series. Vol. 8. London, Hodgson, 1905. 8vo. 6s. 6d.

Mayer (J. E.). Das mathematische Pensum des Primaners. Ein Hilfsbuch für den Primaner humanistischer und realistischer Gymnasien, für Techniker, etc., sowie besonders für das Selbststudium. Heft 19, 20: Elemente der Mechanik. Leipzig, Schäfer, 1905. 8vo. 86 pp. M. 2.00

Moraga y Sánchez (F.). Programa de algebra y trigonometría. Madrid, Ratés, 1905. 8vo. 24 pp. F. 1.00

Newson (H. B.). Graphic algebra for secondary schools. Boston, Ginn, 1905. 12mo. 19 pp. $0.10

Pahl (F.). See Schulze (E.).

Peter (P.). Die Zahlenwelt im Bilde. Eine kleine mathematische Unterhaltung. (Progr.) Feldkirch, 1905. 8vo. 21 pp.

Pierce (R. M.). Problems of number and measure. Outline and bibliography of the application of the Arabic system of notation to a radix other than ten and account of the power of the resulting new system of non-decimal arithmetic as the basis of a throughgoing reform of the metric systems of the world. New York, 1905. 12mo. 21 pp. Cloth. $1.25

Schultz (E.). Vierstellige Logarithmen der Winkelfunktionen. (a) Von 10″ zu 10″. (b) Von 1′ zu 1′. Essen, Baedeker, 1905. 8vo. 42 pp. M. 0.30

Schulze (E.) und Pahl (F.). Mathematische Aufgaben. Ausgabe für Gymnasien. 1ter Teil: Aufgaben aus der Planimetrie und Arithmetik für die Unterstufe. Leipzig, Dürr, 1905. M. 2.40

Smith (D. E.). A modern American course of study in arithmetic arranged by years. Boston, Ginn, 1905. 16mo. 22 pp.

——. Practical arithmetic. Boston, Ginn, 1905. 12mo. 546 pp. Cloth. $0.65

——. Handbook to Smith's arithmetics. Boston, Ginn, 1905. 12mo. 125 pp.

Soulé (G.). Philosophical practical mathematics. New Orleans, 1905. 8vo. 1013 pp. Cloth. $6.00

Steiger (J.). Der geometrische Unterricht in der Volkschule. Zum Gebrauche in Seminarien und für die Hand des Lehrers methodisch dargestellt. 3te, verbesserte Auflage. Bühl, Konkordia, 1905. 6 + 5 + 69 pp. M. 1.00

Stern (H. A.) and Topham (W. H.). Practical mathematics. London, Bell, 1905. 8vo. 388 pp. Cloth. 4s. 6d.

Strenger. Ueber halbregelmässige Vielflache. (Progr.) Schwäb. Hall, 1905. 4to. 45 pp.

Suárez Somonte (I.). Programa de geometría. Madrid, Ratés, 1905. 8vo. 38 pp. Fr. 1.00

——. Programa de nociones y ejercicios de aritmética y geometría. Madrid, Ratés, 1905. 8vo. 36 pp. (Instituto del Cardenal Cisneros.) Fr. 1.00

Topham (W. H.). See Stern (H. A.).

Worked Examples in applied mathematics. (*University Tutorial Series.*) London, Clive, 1905. 8vo. 32 pp. 1s.

Young (G. C. and W. H). The first book of geometry. London, Dent, 1905. 12mo. Cloth.

Zotti (P.). I logaritmi nei conteggi commerciali e bancari: teoria e pratica, ad uso delle scuole commerciali e istituti tecnici. Milano e Roma, Albrighi, 1905. 16mo. 64 pp. L. 0.80

Zwerger (M.). Studien im Gebiete der elementaren Mathematik. (Progr.). Würzburg, 1905. 8vo. 44 pp.

III. APPLIED MATHEMATICS.

Abraham (H.) et Langevin (P.). Les quantités élémentaires d'électricité : ions, électrons, corpuscules. Un volume en deux fascicules. Paris, Gauthier-Villars, 1905. 8vo. 16 + 1138 pp. (Mémoires de la Société de physique, 2e série.) Fr. 35.00

Alger (P. R.). Hydromechanics : an elementary treatise prepared for the use of the midshipmen. Annapolis, Md., Alger, 1905. 12mo. 145 pp. Cloth. $1.50

Bach (C.). Elastizität und Festigkeit. Die für die Technik wichtigsten Sätze und deren erfahrungsmässige Grundlage. Mit in den Text gedruckten Abbildungen und 20 Tafeln in Lichtdruck. 5te, vermehrte Auflage. Berlin, Springer, 1905. 8vo. 24 + 668 pp. Cloth. M. 18.00

Baker (W. M.). Elementary dynamics. 2d edition, revised. London, Bell, 1905. 16mo. 318 pp. Cloth. 4s. 6d.

Bartlett (G. M.). Numerical problems in descriptive geometry for class and drawing-room practice. Ann Arbor, Mich., Bartlett, 1905. 12mo. 62 pp. $0.50

Bauschinger (J.). Die Bahnbestimmung der Himmelskörper. Leipzig, Engelmann, 1906. 8vo. 16 + 653 pp. M. 34.00

Belinfante (A. A. C.). De duwkracht. Beschouwingen van stoffen en hun bewegingen als gevolgen van botsende- of duwkracht ; verwerping der aantrekkingskracht, zwaartekracht en moleculaire krachten. Westzaan, 1905. 8vo. 186 pp. M. 5.00

Bentabol y Ureta (H.). Cuestiones astronómicas y de mecánica celeste. Madrid, 1905. 8vo. Fr. 4.00

Blau (E.). Die Mechanik fester Körper. Lehrbuch in elementarer Darstellung für höhere technische Fachschulen und zum Selbstunterricht, nebst einer Sammlung von 250 aufgelösten Beispielen. Hannover, Jänecke, 1905. 8vo. 7 + 263 pp. M. 6.00

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THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE eighteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Friday and Saturday, December 29 and 30, 1905. Morning and afternoon sessions were held on each day, the first session opening on Friday at 10:30 o'clock. The following members were present:

Professor D. F. Campbell, Dr. E. L. Dodd, Professor J. F. Downey, Professor J. W. Glover, Professor A. G. Hall, Professor Thomas F. Holgate, Dr. L. C. Karpinski, Professor Kurt Laves, Mr. N. J. Lennes, Dr. A. C. Lunn, Professor E. H. Moore, Professor H. B. Newson, Dr. J. C. Morehead, Miss Ida M. Schottenfels, Professor E. B. Skinner, Professor H. E. Slaught, Professor E. J. Townsend, Mr. R. E. Wilson, Professor J. W. Young.

Professor A. G. Hall was elected chairman of the meeting. This being the regular time for the election of officers, the Section directed the nominating committee to report an executive committee for the Section to consist of a chairman, a secretary, and one additional member; this committee to have charge of all business of the Section including the preparation of the programme.

The officers elected for the year 1906 were as follows: Chairman, Professor Alexander Ziwet; Secretary, Professor H. E. Slaught; additional member of the executive committee, Professor A. G. Hall.

By vote the Section expressed its appreciation of the services of Professor Thomas F. Holgate, the retiring secretary, who had served the Section since its organization.

The executive committee, together with Professors Townsend and Skinner, were constituted a committee to arrange at an early meeting for a full half day conference of the Section with the deans and professors of engineering in technical schools, on the teaching of mathematics to engineering students.

The executive committee was requested to arrange so that the Section may take up the study of questions relating to applied mathematics, to the teaching and history of mathematics, and to the establishment of closer relations with colleagues in engineering and other professional subjects.

On the evening of Friday a dinner was held in the Hotel Del Prado, at which nineteen persons were present.

The following papers were read at this meeting:

(1) Dr. H. L. COAR: "Functions of three real variables."

(2) Dr. E. L. DODD: "A theorem concerning implicit functions, and its geometric aspect."

(3) Professor KURT LAVES: "A dynamical interpretation of an integral of Jacobi's partial differential equation for the problem of a solid body rotating about a fixed point."

(4) Professor J. W. YOUNG: "On discontinuous groups defined by the normal curve of the fourth order in a space of four dimensions."

(5) Professor E. H. MOORE: "On the theory of systems of integral equations of the second kind" (preliminary communication).

(6) Professor J. W. GLOVER: "The teaching of actuarial theory in universities and colleges."

(7) Professor H. B. NEWSON: "Complex number systems and continuous groups" (preliminary communication).

(8) Dr. A. C. LUNN: "Sets of postulates for the trigonometric functions."

(9) Dr. J. C. MOREHEAD: "Numbers of the forms $2^kq \pm 1$."

(10) Mr. N. J. LENNES: "Curves in non-metrical analysis situs."

(11) Mr. N. J. LENNES: "A fundamental existence theorem in the calculus of variations."

(12) Mr. N. J. LENNES: "Note on the Heine-Borel theorem."

(13) Dr. A. E. YOUNG: "On certain isothermic surfaces."

(14) Professor J. B. SHAW: "Canonical forms of hypercomplex numbers" (preliminary communication).

(15) Professor JACOB WESTLUND: "Primitive roots of an ideal in an algebraic number field."

(16) Professor L. E. DICKSON: "On quadratic, hermitian, and bilinear forms."

(17) Professor L. E. DICKSON: "Expressions for the elements of a determinant in terms of the minors of a given order. Generalization of a theorem due to Studnička."

(18) Professor G. A. MILLER: "The groups of order p^m which contain exactly p cyclic subgroups of order p^α."

Dr. Coar was introduced by Professor Townsend. In the absence of the authors, the papers by Dr. Young, Professor Shaw, Professor Westlund, Professor Dickson, and Professor Miller were read by title.

Professor H. B. Newson introduced for general discussion the question of "The distribution of mathematical subjects during the first two years of the college course." This discussion was participated in generally by members of the Section and others present. The main question was the best possible arrangement of the work for these two years on the assumption that the student enters with algebra through the progressions and with plane and solid geometry. Professor Newson stated the problem before the state universities of the middle west, where in most places college and engineering students are taught in the same classes. The basis of the discussion was the new programme at the University of Kansas, and the opinion prevailed that college and engineering students should be given the same courses.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Coar spoke on functions of three real variables which are completely defined and single-valued for all values of the variables. In the case of such functions three kinds of limits exist: (1) the triple limit $\lim_{x \doteq x_0} \lim_{y \doteq y_0} \lim_{z \doteq z_0} f(x, y, z)$; (2) the double simultaneous limit $\lim_{x \doteq x_0} \lim_{y \doteq y_0, z \doteq z_0} f(x, y, z)$; (3) the simultaneous limit in three variables $\lim_{x \doteq x_0,\, y \doteq y_0,\, z \doteq z_0} f(x, y, z)$. Necessary and sufficient conditions for the existence of each of these were given. The conditions for the interchange of the order of sequence of these limits, of which there are six cases, were obtained, and questions of the continuity and discontinuity of the limiting functions were discussed. The function may converge uniformly to the limiting function in two ways, either with respect to one variable or with respect to two variables taken together. The existence of either of these cases involves that of the other. The adaptability of the ideas of limits to functions defined by a series of functions of two real variables was shown, as was the bearing of this on the problem of term by term integration in the case of simultaneous integrals.

2. Dr. Dodd pointed out that it is intuitionally evident that a continuous surface, passing through a point (a, b) of the (x, y)-

plane, will cut the plane in a continuous line through (a, b), provided the surface inclines in the same general direction near (a, b). This suggests the following theorem regarding implicit functions, a theorem which may be proved analytically:

If $f(x, y)$ is a continuous function of x and y, and a univariant function of y near (a, b), where f vanishes, then there will exist one and only one continuous function $y = \phi(x)$ such that $\phi(a) = b$ and $f[x, \phi(x)] \equiv 0$ when x is near a.

The hypothesis does not require that $\partial f/\partial y \neq 0$, nor even that this derivative should exist. The theorem can be generalized.

3. After Poisson had applied Lagrange's method of variation of constants to the problem in question, it was Richelot who, in 1850, obtained the same final differential equations of perturbed motion by applying to it the Hamilton-Jacobi method. The difficult task of finding a complete solution of Jacobi's partial differential equation necessitated very laborious transformations. By the introduction of a certain spherical triangle Radau brought about a simplification of the method. In Professor Laves's paper the attempt is made to accentuate the geometric aspects of the problem by giving to the integral under investigation a dynamical interpretation which seems to have escaped the attention of those who have worked on the problem.

4. Fricke has defined arithmetically an extensive class of discontinuous groups of linear fractional substitutions of the form

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \qquad (\alpha\delta - \beta\gamma = 1)$$

as the reproducing groups of ternary quadratic forms. Professor Young's paper aims at a generalization of Fricke's procedure. The conic represented by the ternary quadratic form is replaced by a normal curve C_n of order n in a space S_n of n dimensions. This curve is transformed into itself by a group of ∞^3 collineations in S_n, each collineation subjecting the points of C_n, which depend rationally on a parameter ζ, to a substitution of the form $\zeta' = (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$. If the coefficients in the collineations are restricted to be integers with determinant unity, the collineation groups on the z_i, the point coördinates in S_n, and the holoedrically isomorphic group on the parameter ζ will be discontinuous. Thus every C_n defines a discontinuous ζ-group.

Professor Young first shows that the ζ-groups defined as above by two normal C_n's that can be transformed into each other by a rational collineation (*i. e.*, one with rational coefficients) are commensurable, *i. e.*, they have a subgroup of finite index in common; for this reason it is natural to confine oneself to the groups defined by C_n's obtained from a canonical form by irrational collineations. The author then restricts himself to the case $n = 4$, and in particular to C_4's obtained from the canonical form: $z_1 = \zeta^4$, $z_2 = \zeta^3$, $z_3 = \zeta^2$, $z_4 = \zeta$, $z_5 = 1$ by transformations of the form:

$$z_1' = \sqrt{p}z_1 + \sqrt{q}z_5, \qquad z_2' = \sqrt{m}z_2 + \sqrt{n}z_4, \qquad z_3' = \sqrt{r}z_3,$$
$$z_4' = \sqrt{m}z_2 - \sqrt{n}z_4, \qquad z_5' = \sqrt{p}z_1 - \sqrt{q}z_5,$$

where p, q, m, n, r are positive integers. No essential restriction is involved in supposing that none of these integers contains a square factor and that p is prime to each of the remaining four. It is then found that the ζ-group $[p, q, m, n, r]$ will exist (*i. e.*, consist of more than the identical substitution) only if $r = p$, and further that this group is essentially the same as the reproducing group of the ternary quadratic form $Pz_1^2 - Qz_2^2 - Rz_3^2$, where P is the highest common factor of q and m, Q the highest common factor of m and n, and R the highest common factor of q and n.

5. Denoting (1) by R a range of values of arguments or indices s, t; (2) by F a field of ordinary complex numbers; (3) by Γ_{1s} a class or category of single-valued functions f_s of s over R, with functional values belonging to F; (4) by Γ_{2st} a category of single-valued functions K_{st} of (s, t) over (R, R), with functional values belonging to F; (5) by $\overparen{t}$ an operation applicable to functions of Γ_{1t} and serving to eliminate t from such functions, — Professor Moore studies postulates involving the symbols $(R, F, \Gamma_1, \Gamma_2, \frown)$ in such a way as to yield a theory of the linear equation of the second kind,

$$g_s = f_s + \overparen{K_{st}^{t} f_t},$$

holding as s ranges over R, a theory which suitably specialized becomes a theory of the system of linear integral equations of the second kind,

$$g_i(x) = f_i(x) + \sum_{j=1}^{m} \int_a^b K_{ij}(x, y) f_j(y) dy \qquad \left({i=1, \ldots, n \atop x=a, \ldots, b} \right).$$

8. Dr. Lunn's paper gave several sets of independent postulates which, with the accompanying definitions, sufficed to determine the sine, cosine, and tangent, as real functions of a real variable.

9. Dr. Morehead's paper is supplementary to his paper read at the April, 1905, meeting of the Society, on numbers of the forms $2^k q \pm 1$ and Fermat's numbers. For the purpose of determining the primes and the prime factors of non-primes of the forms $2^k q \pm 1$, two positive functions, $q_k(p)$, $q_k'(p)$, of the odd prime p were defined. These functions possess the properties: $2^k q + 1 \equiv 0 \bmod p$, for $q = np + q_k$. $2^k q' - 1 \equiv 0 \bmod p$, for $q' = np + q_k' (n = 0, 1, 2, \cdots)$ and conversely. Thus for $q_{2^n} = 1$ we have $2^{2^n} \cdot 1 + 1 \equiv 0 \bmod p$, *i. e.*, p is a factor of the Fermat number $F_n = 2^{2^n} + 1$. From the various properties of q_k, q_k' is developed a means of computing q_{2^n}, which involves in general less labor than the usual method of computing successively the residues with respect to p of 2^{2^1}, 2^{2^2}, 2^{2^3}, $\cdots$ 2^{2^n} in order to determine whether or not p is a factor of F_n.

The second part of the paper gives several theorems on the quadratic characters of q_k, q_k' with respect to the corresponding prime p, and shows how some, and often all, of the quadratic residues and non-residues of p can be written down at once.

10. In the first part of this paper Mr. Lennes gives fresh proofs of classical theorems on the separation of the plane by simple finite and infinite polygons and by a simply closed Jordan curve. The frontier of a region is considered and a set of conditions are specified under which such frontier is a Jordan curve. The definition of a continuous curve is at once translated into non-metrical language and the treatment is non-metrical throughout. In the second part of the paper several definitions of a continuous curve are given, one of which is developed as follows:

A set of points is connected if in every pair of complementary subsets at least one subset contains a limit point of points in the other set.

A set $\{P'''\}$ is said to separate a connected set $\{P\}$ into two sets $\{P'\}$ and $\{P''\}$ if every connected set lying in $\{P\}$ and containing points of both $\{P'\}$ and $\{P''\}$ also contains at least one point of $\{P'''\}$.

A closed, bounded, connected set of points containing the points A and B is a simple arc connecting A and B if every point of the set, except A and B, separates it into two sets, one of which contains A and the other B. Two simple arcs connecting A and B and having no point in common except A and B constitute a Jordan curve.

This definition of continuous curves applies equally well to curves in an n-dimensional manifold as to curves in the plane.

11. In this paper Mr. Lennes considers a set of arcs of Jordan curves connecting a fixed pair of points A and B. The set of arcs lies in a closed region R, having a connected interior. The expression "entirely open region" is used to denote a set of points, no point of which is a limit point of points not in the set. An entirely open region of the region R is a set of points of the region R no part of which contains a limit point of points of R which are not of the set.

In the region R is considered the set $[C_{AB}]$ of all arcs connecting A and B which satisfies the following condition :

For every point P of the region R and for every triangle t of which P is an interior point there exists a triangle t' of which P is an interior point such that no arc of the set $[C_{AB}]$ contains a point on t between two of its points on t'. (This may be regarded as a uniform continuity assumption on the set $[C_{AB}]$.)

Associated with each arc C_{AB} of the set $[C_{AB}]$ is a positive number $M(C_{AB})$ forming a set of numbers $[M(C_{AB})]$ corresponding to the set of arcs $[C_{AB}]$. The set of numbers $[M(C_{AB})]$ is assumed to satisfy the following condition : For every given arc C'_{AB} of the set $[C_{AB}]$ and for every given positive number ϵ, there exists an entirely open region R' of R, such that for every arc C''_{AB} of $[C_{AB}]$ and of R', $M(C'_{AB}) - M(C''_{AB}) < \epsilon$. With these assumptions it is proved that there exists at least one arc $C^{(m)}_{AB}$, of the set $[C_{AB}]$ such that $M(C^{(m)}_{AB})$ is the greatest lower bound of the set $[M(C_{AB})]$.

12. In this note Mr. Lennes formulates the Heine-Borel theorem on non-metrical hypotheses, as follows :

If in a three-dimensional continuous manifold every point of a closed set $[P]$ is an interior point (fails to be a limit point of points not of the set $[R]$) of at least one set of points $[R]$ of a set of sets $[[R]]$, then there exists a finite subset $[R]_1$,

$[R]_2, \cdots, [R]_n$ of the set $[[R]]$, such that every point of $[P]$ is an interior point of at least one of the sets $[R]_1, \cdots, [R]_n$.

The following corollary is a useful form for metrical analysis:

If every point of $[P]$ lies within at least one of the sets of $[[R]]$ then there exists a positive number δ such that every cube whose edge is δ and whose center is a point of the set $[P]$ lies within at least one of the sets of $[[R]]$.

13. In a paper presented at the sixteenth meeting of the Chicago Section (December, 1905), Dr. Young gave the result of a certain investigation leading to the consideration of some new isothermic surfaces, which, when referred to lines of curvature, have their linear element in the form $ds^2 = (u + v)^n(du^2/U + dv^2/V)$, where n is a constant and U and V are functions of u and v, respectively.

In the present paper he discusses the problem of determining all surfaces corresponding to the linear element above when referred to lines of curvature. Two distinct cases have to be considered, the one where $n = \pm 2$, and the other where n has any other value. In the latter case Dr. Young shows first that the most general forms of the functions U and V are Laurent's series in u and v, respectively. A further discussion of this case shows that if the functions U and V are assumed to be analytic, then $n = \pm 1$ and U and V take very simple forms. When $n = 1$, one form of the functions U and V gives the sphere, while another form gives the other quadric surfaces. Likewise, when $n = -1$, a certain minimal surface corresponds to the first form of the functions U and V and certain other surfaces peculiarly related to the other quadrics correspond to the second.

The surfaces corresponding to the case $n = -2$ are the cyclides of Dupin and certain other envelopes of spheres, whereas the surfaces corresponding to $n = +2$ are the new surfaces which were considered in the previous paper.

This discussion is of particular interest in that it gives a good example of the comparative ease with which positive results may be obtained by working from the Gauss and Codazzi equations. All the quadratures are performed and the cartesian coördinates of all the new surfaces are given as functions of u and v.

14. Professor Shaw's paper develops the theorem that any

number in any linear associative algebra can be written in the form

$$\phi = \sum_{i,j} A_{i,j}^{(s\cdots\mu_i)} \eta_{i,j} \qquad (i, j = 1, \cdots, \rho),$$

where $\eta_{i,j}$ are quadrate vids, subject, that is, to the law $\eta_{ij}\eta_{jk} = \eta_{ik}$; $\eta_{ij}\eta_{kl} = 0$ if $j \neq k$. The quantities $A_{i,j}^{(s\cdots\mu_i)}$ are polynomials in a nilpotent ν, such that the power $\nu^{\mu_1+1} = 0$; the coefficients of the powers of ν are from any field, scalar, abstract, etc. The powers of ν in $A^{(s\cdots\mu_i)}$ start with the sth and run consecutively to the μ_ith. The number s is such that $s = 0$ if $\mu_i \leqq \mu_j$, $s = \mu_i - \mu_j + 1$ if $\mu_i > \mu_j$. The numbers μ_i $(i = 1, \cdots, \rho)$ are integers assigned beforehand and are such that $\mu_1 \geqq \mu_2 \geqq \mu_3 \cdots \geqq \mu_\rho$.

The consequences of this theorem are developed partially; in particular, for Peirce algebras, that is, non-quaternion algebras with no skew units. Such algebras are generated by ρ linearly independent, but not necessarily productly independent units. When their polynomials are known the algebra is determined. The persistent coefficients of these polynomials under linear transformation determine the different algebras of the same type, in terms of their "natural units." This paper is a continuation of "Theory of linear associative algebra" and "Nilpotent algebras" by the author (*Transactions*, Volume 4, 1903).

15. Professor Westlund's paper is a continuation and extension of a former paper. It is shown that in an arbitrary algebraic number field the only ideals which have primitive roots are of the form P^n, $Q_1^{n_1} Q_2 \cdots Q_i$, and $Q_1 Q_2 \cdots Q_i P^n$, where p, the rational prime divisible by P, is odd, and Q_1, Q_2, $\cdots$ are the prime ideal factors of 2. A method is also given for determining the primitive roots in each case.

16. The first paper by Professor Dickson treats of the reduction to canonical types of quadratic and hermitian forms in a general field. Application is made to quadratic and bilinear forms invariant under a given substitution S and their normalization by substitutions commutative with S. The latter problem is a generalization to an arbitrary finite or infinite field of the recent memoir by Jordan in the *Journal de Mathématiques* on the case of a finite field of prime order. The paper will appear in the *Transactions*, April, 1906.

17. The second paper by Professor Dickson gives a practical method of expressing the elements of a determinant D of order n as rational functions of the minors M_m of order m and the mth root of a rational function of the M_m. If from $m - 1$ rows of D we build $n - m$ minors $M^{(i)}_{m-1}$ such that the elements of M' are chosen from any columns other than a fixed column C_{j_1}, the elements of M'' from any columns other than C_{j_1} and C_{j_2}, etc., we can express $M'M'' \cdots M^{(n-m)}D$ as a determinant of order $n - m + 1$ whose elements are minors M_m of D. For the case $M' = M'' = \cdots = M^{(n-m)}$ the theorem is due to Studnicka. The latter case enables us to derive the minors M_{m-1} from the M_m, then the M_{m-2} from the M_{m-1}, and finally the elements from the M_2. This symmetric solution is valueless since it employs a complicated array of radicals of various degrees. The other solutions given involve a single irrationality, viz., the mth root of a rational function of the M_m. The paper appeared in the *American Mathematical Monthly*, December, 1905.

18. Professor Miller considers the possible types of groups G of order p^m (p any prime) which contain just p cyclic subgroups of a given order p^α ($\alpha < m$). He finds that every group G of this kind must contain a cyclic subgroup of order p^{m-2}. If p is restricted to be odd and $m > 4$, a stronger theorem is obtained as follows: When p is odd and $m > 4$, G contains exactly p cyclic subgroups of every order from p^2 to p^{m-1}. This theorem is also true when $m = 3$; and when $m = 4$ and $p > 3$. As all the groups of order p^m which contain a cyclic subgroup of order p^{m-2} are known, these results give a complete determination of all the groups of order p^m, which contain exactly p cyclic subgroups of the same order.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILLINOIS,
January 16, 1906.

THE GROUPS CONTAINING THIRTEEN OPERATORS OF ORDER TWO.

BY PROFESSOR G. A. MILLER

(Read before the American Mathematical Society, February 24, 1906.)

It is known that a group of order 2^m cannot contain exactly thirteen operators of order 2, but that there are other groups which have this property.* Let G represent such a group. We shall first consider the case where the operators of order 2 in G form a single set of conjugates. In this case G transforms these operators according to a transitive substitution group of degree 13. The subgroup of G which corresponds to identity in this transitive substitution group will be represented by H. If H were of even order, it would contain an operator of order 2 which would be commutative with every operator of this order contained in G. From this it would follow that each of these operators would be commutative with all of them. This is impossible since they cannot all be contained in a group of order 2^m. Hence *H is of odd order.*

The substitution group which is simply isomorphic with G/H must therefore contain substitutions of order 2. Since the group of order $2h$, h being the order of H, which corresponds to the group generated by such a substitution contains at least one operator of order 2, the substitution group of degree 13 contains exactly 13 substitutions of order 2; hence it involves only one subgroup of order 13 and its order is a divisor of 156. Moreover, *all the operators of H are commutative with every operator of order 2 contained in G.*

The product of two operators of order 2 contained in G corresponds to an operator of order 13 in the isomorphic group. Since this product is transformed into its inverse by each of its factors,† and since the operators of H are commutative with each of these factors, it follows that this product is of order 13. That is, *any two operators of order 2 contained in G generate the dihedral rotation group of order 26.* As each of the operators of this group is commutative with every operator of H, it follows that G contains the direct product of a group of odd order and the dihedral rotation group of order 26 as an invariant subgroup. The quotient group of G with respect to this invariant subgroup is contained in the cyclic group of order 6.

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* BULLETIN, vol. 12 (1905), p. 74.

† BULLETIN, vol. 7 (1901), p. 424.

The order of G cannot be divisible by eight when all the operators of order 2 are conjugate. If this order is divisible by four the subgroups of order 4 are cyclic. Hence it follows that each of the Sylow subgroups of G whose order is a power of two contains only one operator of order 2 when the operators of this order in G form a single conjugate set. Conversely all the operators of order 2 must be conjugate under G whenever a Sylow subgroup of order 2^m contains only one operator of this order. Hence it follows that in the groups which remain to be considered each Sylow subgroup of order 2^m contains more than one operator of order two.

§ 1. *Groups Containing a Set of two Conjugate Operators of Order* 2.

Let s_1, s_2 be two operators of order 2 contained in G, such that s_1 is transformed into s_2 by exactly half the operators of G. As every operator which transforms s_1 into s_2 must also transform s_2 into s_1, G contains operators whose orders are powers of 2 which transform s_1 into s_2. Hence the Sylow subgroups of order 2^m contained in G are non-abelian and their orders must be divisible by 8. As the product s_1s_2 is invariant under G, it is included in every Sylow subgroup of order 2^m. The Sylow subgroups which contain s_1 must therefore also contain s_2. That is, *every Sylow subgroup of order* 2^m *includes the four group generated by* s_1, s_2.

Every Sylow subgroup of order 2^m includes other operators of order 2. Let s_3 represent such an operator. We shall first prove that s_3 cannot be commutative with s_1 and s_2. If this were the case, a Sylow subgroup S would contain at least seven commutative operators of order 2. It could not contain exactly seven operators of order 2, since 10 is not divisible by 4. It could not contain eleven operators of this order;* for there would be some other Sylow subgroup S_1 which would contain at least two operators of order 2 which are not in S. One such operator would transform S into S_2 having at least two operators of order 2 which are not in S. As S_2 could not contain this transforming operator, it follows that S_1 would contain at least two operators which are neither in S nor in S_2. Since S_2 would contain at least two other operators of order 2 which are not in S, it is clear that S could not contain more than nine operators of order 2.

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* This is a special case of the theorem: if a Sylow subgroup of any group G contains more than one subgroup of order p without including all the operators of order p in G, then G contains at least p^2 subgroups of order p which are not in a given Sylow subgroup.

If S contained nine operators of order 2 and if s_3 were also commutative with s_1 and s_2, the operators of order 2 which are not in the group generated by s_1, s_2, s_3 could not be commutative with s_1. Hence s_1, s_2, s_3 would generate an invariant subgroup of S. As an operator of order 2 could not transform this subgroup into itself without being commutative with at least four of its operators, S could not contain exactly nine operators of order 2. Hence, as stated above, s_3 cannot be commutative with s_1 and s_2.

Since s_3 transforms s_1 and s_2, it, together with these two operators, generates the octic group. We shall now consider the case where S contains no operator of order 2 besides those of this octic group. As all the conjugates of S have three operators in common, there are just five distinct octic subgroups in these conjugates. These subgroups are transformed by all the operators of G according to a transitive group T of degree 5. Since T contains a complete set of five conjugate operators of order 2, it can involve only one subgroup of order 5. The operators of G which correspond to identity of T constitute an invariant subgroup H of G. The operators of H which are commutative with s_3 constitute a subgroup of half the order of H. This subgroup H_1 includes s_1, s_2.

As H_1 contains only one operator of order 2, its Sylow subgroups of order 2^m either are cyclic or they contain cyclic subgroups of half their order while the remaining operators are of order 4. They must therefore contain an odd number of cyclic subgroups of order 4 if they contain one such subgroup. If the order of H_1 were divisible by 4, H would contain an operator of order 4 which would be transformed into its inverse either by s_1 or by s_3. Hence either H would contain more than three operators of order 2, or a divisor of G corresponding to an operator of order 2 in T would contain more than two such operators. As this is impossible, *H contains only one subgroup of order 4, viz., the one generated by s_1, s_2, and the order of H is not divisible by 8.*

Two operators of order 2 in G which correspond to two distinct operators of order 2 in T have for their product an operator whose order is divisible by 5 but by no other odd prime, since s_3 is commutative with all the operators of H_1. The order of this product could not be divisible by 4 since H does not include any operator of this order. Moreover, the two given factors may be so selected as to make the order of this product either 5 or 10. In the latter case they generate the dihedral rotation group of order 20. This group contains all the operators of order 2 in G except s_1 and s_2. These two operators are commutative with the operators of its cyclic subgroup of order 10, but they transform the remaining operators into themselves multiplied by s_1s_2. Hence *the operators of order* 2 *in* G *generate the group of order* 40 *which contains* 13 *operators of order* 2.

As the order of H_1 is not divisible by 4 and as H_1 contains only one operator of order 2, it is the direct product of a group of odd order and s_1s_2. This group of odd order is clearly invariant under G. In fact, H is the direct product of this group and the group generated by s_1, s_2. Hence G contains the direct product of a group of odd order and the group of order 40 which contains exactly thirteen operators of order 2. If G contained any other operators, it would contain this direct product as an invariant subgroup of half its order, and the remaining operators would correspond to operators of order 4 in T. The square of such an operator would therefore transform s_1 into s_2, since s_3 has this property. This is clearly impossible as s_1 has only two conjugates under G.

We have now proved that G is the direct product of a group of odd order and the group of order 40 which contains just thirteen operators of order 2, whenever s_3 transforms s_1 into s_2 and a Sylow subgroup contains no operators of order 2 besides those contained in the octic group generated by s_1, s_2, s_3. Moreover, the direct product of *any* group of odd order and this group of order 40 possesses the properties in question. It remains to consider the case when S contains more than one octic subgroup involving s_1, s_2. Each of these octic subgroups is transformed into itself by every operator of order 2 contained in S. There could not be more than three such octic subgroups involving s_1, s_2, since S cannot involve eleven operators of order 2. If there were three, one would be invariant under S. The operators of this subgroup would transform each of the others into itself. Hence each would be transformed into itself by the other two, and the three would generate a group whose commutator subgroup would be 1, s_1s_2. As such a group would contain more than nine operators of order 2, S cannot contain more than two octic subgroups involving s_1, s_2.

As one octic group would transform one other octic group into itself, it would transform two of the five contained in G into themselves. Hence G would have to transform the five octic subgroups according to a transitive group of degree 5 containing a transposition. This is clearly impossible. Hence *G contains a set of two conjugate operators of order 2 only when it is the direct product of the group of order 40 which contains thirteen operators of order 2 and some group of odd order.*

§2. *Proof that there is no Group in which the Number of Operators of Order 2 in a Complete Set of Conjugates is Three, Four or Five.*

If there were a set of three operators of order 2 which formed a complete set of conjugates under G, they would be transformed according to a group of degree 3 by all the operators of G. The subgroup H which would correspond to the identity in this group of degree 3 could not involve these three conjugates; for, if they were all contained in a four group, H could not contain the thirteen operators of order 2 since $13 \not\equiv 3 \bmod 4$. Neither could H contain just seven operators of this order, since they would be included in an invariant subgroup of order 8 and hence G would have to involve more than thirteen operators of order 2. Since the number of operators of order 2 which are in G without being also in H is divisible by 6, no other numbers require consideration. If the three given operators generated a group of order 8, this would be invariant, and hence this case is impossible. If the three conjugate operators in question were not in H, there would be just one more operator of order 2 in a divisor corresponding to a substitution of order 2 than in H. This is again impossible because $13 \not\equiv 3 \bmod 4$. Hence there cannot be a complete set of three conjugate operators of order 2 under G.

If G contained a complete set of four conjugate operators of order 2, at least two of these would be in a four group. If three of them were in this group, they would all be commutative and hence would generate an invariant subgroup of order 8. As this is impossible, the four group which contained two of these four conjugates would not contain any other. It is easy to see that this four group would be transformed into itself by the other conjugates and that the only case which requires further consideration is when these four conjugates generate an invariant octic group, and when S contains nine operators of order 2. No two Sylow subgroups could have only five common operators of order 2, since one of the remaining operators of order 2 in one of these Sylow subgroups could not transform the other into itself and hence would transform it into one having additional operators of order 2. As this is impossible, any two Sylow subgroups would have at least seven common operators of order 2.

The eight operators of order 2 which are not in the octic subgroup generated by the four conjugate operators in question would therefore be divisible into four pairs such that each pair with this octic group would generate a subgroup involving just seven operators of order 2. As the number of subgroups involving nine operators of order 2 would be odd, and as all would be conjugate under G, this is impossible. The impossibility of constructing such a group G can also be proved by observing that the nine operators of order 2 in S would generate a group of order 32 which would have a commutator subgroup of order 2 and contain the given octic group as invariant subgroup. As such a group cannot be constructed, it is proved that G could not contain a complete set of four conjugate operators of order 2.

If G contained a complete set of five conjugate operators of order 2, it would transform them according to a transitive substitution group of degree 5. Let H represent the subgroup of G corresponding to identity in this transitive substitution group. As H cannot involve the five conjugate operators of order 2, it contains just three operators of this order. This is impossible, since each of these three operators would be commutative with each one of the five conjugate operators of order 2, and hence each divisor corresponding to a substitution of order 2 would contain just four operators of order 2.

§ 3. *Groups in which there is a Complete Set of Six Conjugates of Order* 2.

If G contains a complete set of six conjugate operators of order 2 it must transform them according to a transitive substitution group T of degree 6. The conjugate operators of order 2 cannot be contained in the subgroup H which corresponds to identity of T, since G cannot contain an invariant subgroup of order 8 which involves no operator of order 4. If these six conjugates correspond to the same substitution of T, H would contain five operators of order 2. The remaining two operators could not correspond to this substitution of T. Hence they would correspond to another invariant substitution. This is impossible since T cannot contain two invariant substitutions of order 2. As every non-invariant substitution of order 2 has at least three conjugates under T, the six conjugate operators in question correspond either to three or to six substitutions of T.

In the latter case H would be of odd order; for, if it were of even order the operators of order 2 which would correspond to two commutative conjugate substitutions of degree < 6 could not be commutative since their product could not be of order 2. That is, there would not be a complete set of six conjugates of order 2 in G. If H were of odd order, only one operator of G would correspond to each of the six conjugates in question, and at least one operator of order 2 would correspond to every substitution of order 2 in T. It follows directly from the properties of the T's which have six conjugates of order 2 that this is impossible, hence the six conjugate operators of order 2 in question correspond to three conjugate substitutions whose degree is < 6, and H involves only one operator of order 2. If these three substitutions of T were commutative, the corresponding operators of order 2 would also be commutative, or else they would generate a group whose commutator subgroup is of order 2.

The former of these two alternatives is clearly impossible. In the latter, the six conjugates in question would generate a subgroup of order 16, and a Sylow subgroup of order 2^m would involve at least 11 operators of order 2. As this is impossible, it follows that the three conjugate substitutions of T which correspond to the six conjugates in question must generate a group of order 6. These conjugates must therefore generate the dihedral rotation group of order 12, and hence this is an invariant subgroup of G, and T contains an invariant subgroup of order 3. From this it follows that T is one of the two groups of order 12.

We shall now prove that the order of H cannot be divisible by 4. If it were divisible by 4, a Sylow subgroup of H would contain an odd number of cyclic subgroups of order 4. Hence one of them would be invariant under S. If an operator of order 2 not included among the six conjugates in question transformed the generator of this invariant subgroup into its third power, it would correspond to the invariant substitution of T and together with the given generator would generate the octic group. The two operators of order 2 which would be in the same divisor but not in this octic group would transform this octic group into itself. Hence this octic group would be invariant under the group generated by all the operators of order 2 in G. As the operators of G which would correspond to the invariant substitution of T without being in this octic group could not transform the operator of order 4 in this octic group into its inverse they would be commutative with it, and hence this divisor would contain operators of order 4. Such operators would be transformed into their inverse by some operator belonging to the six conjugates in question. As this would lead to operators of order 2 in a divisor corresponding to another substitution of T, it is impossible.

If an operator of order 2 corresponding to the invariant substitution of order 2 in T were commutative with the given operator of order 4, it would transform an operator of order 4 corresponding to one of the six conjugates in question into its inverse. As this is clearly impossible, it is proved that H could not involve an operator of order 4 in case an operator of order 2 in G corresponds to an invariant substitution in T. We shall now prove that the order of H is not divisible by 4 when each of the operators of order 2 in T corresponds to a non-invariant substitution in T. If H contained operators of order 4, such operators would correspond to each non-invariant substitution of order 2 in T. From this it follows directly that there would be operators of order 2 in the divisor corresponding to the invariant substitution of T. As this is impossible, it is proved that *the order of H is not divisible by* 4.

If the operators of order 2 which are not in the given set of six conjugates correspond to non-invariant substitutions of T, two operators of order 2 can be found such that their product is of order 12, since no operator of order 2 corresponds to the invariant substitution in T. Hence the operators of order 2 in G generate the dihedral rotation group of order 24, and G s the direct product of this group and the largest subgroup of odd order contained in H. Moreover, every direct product of this dihedral rotation group and an arbitrary group of odd order contains just thirteen operators of order 2 which have the property in question.

If six operators of order 2 correspond to the invariant substitution of order 2 in T, they generate the dihedral rotation group of order 12, and H contains an invariant subgroup of order 3. As the largest subgroup of odd order in H transforms these six operators according to a group of odd order which has two transitive constituents of degree 3, it must contain an invariant subgroup of one-third its order which does not include the operator of order 3 in the said dihedral rotation group of order 12. Hence H is the direct product of a subgroup of order 2, a subgroup of order 3, and some group of odd order. Moreover, G contains the direct product of the dihedral rotation group of order 12, generated by the six operators of order 2 which are transformed according to T, and the invariant subgroup of order 3 in H.

The six operators of order 2 which correspond to the invariant substitution of T transform this direct product of order 36 into itself and generate with it the group of order 72 which contains exactly 13 operators of order 2 and is generated by these operators. This subgroup of order 72 is invariant in G and has only identity in common with the given group of odd order in H. Hence G is the direct product of some group of odd order and this group of order 72 whenever some of its operators of order 2 correspond to the invariant substitution of T, and every such direct product satisfies the condition in question. The result of this section may be expressed as follows: *Whenever G contains six operators of order 2 which constitute a complete set of conjugates, its thirteen operators of order 2 generate either a group of order 24 or a group of order 72 and G is the direct product of one of these groups and some group of odd order.* Every such direct product contains exactly six conjugates of order 2, and involves 13 operators of this order.

§ 4. *Proof that there is no Additional Group in which the Number of Operators of Order 2 in a Complete Set of Conjugates is Seven, Eight, Nine, Ten, or Eleven.*

If there were a complete set of either seven, eight or nine conjugates of order 2, the remaining operators of this order could not be invariant, since G cannot contain the group o order 8 which contains 7 operators of order 2 as an invariant subgroup. These remaining operators could not occur in sets of conjugates, as these sets have been considered above. If G contained three invariant operators of order 2, while the remaining 10 formed a single set of conjugates, the three invariant operators would generate the four group. As any one of the remaining operators of order 2 and this four group would generate the group of order 8 which has 7 operators of order 2, the remaining operators could be arranged in distinct sets of four operators, which is evidently impossible.

If G contained two invariant operators, their product would also be invariant and hence there could not be a complete set of eleven conjugates. It only remains therefore to consider the case where G contains a single invariant operator of order 2 while the remaining 12 constitute a complete set of conjugates. It will be found that there are groups which come under this case. Hence there are groups containing exactly 13 operators of order 2 in which these operators form sets of conjugates containing any of the following numbers of operators: 13; 1, 2, 10; 1, 6, 6; 1, 12.

§ 5. *Groups in which there is a Complete Set of Twelve Conjugates of Order* 2.

Since such a G contains an invariant operator of order 2, it contains six conjugate four groups and transforms them according to a transitive substitution group T of degree 6. If the operators H of G which correspond to the identity of T included more than one operator of order 2, these six subgroups of order 4 would generate a group of order 2^m involving thirteen operators of order 2. As this is impossible, T contains a complete set of either three or six conjugate substitutions of order 2 and of degree < 6, corresponding to the operators of order 2 in G.

In the former case these three conjugate substitutions of T cannot be commutative, since all the operators of order 2 in G cannot be contained in a Sylow subgroup. Hence T is the group of order 12 which contains seven substitutions of order 2. The four operators of G which correspond to the same substitution in T generate the octic group. As the operators of order 2 in the other divisions transform this octic group into its conjugates, these three conjugates have the cyclic subgroup of order 4 in H in common. Hence this cyclic subgroup is invariant under G. From this it follows directly that H cannot contain any operator of order 8; for if H contained such an operator, its square would generate the only invariant cyclic subgroup of order 4, but this square could not be in the given octic groups.

We shall now prove that the Sylow subgroups of H could not include the quaternion group. If this group occurred in H, a Sylow subgroup S of G would be of order 32 and would include two non-cyclic subgroups of order 16 which would involve only one operator of order 2. As it would contain another subgroup of order 16 involving a cyclic subgroup of order 8, it would contain just three cyclic subgroups of order 8. This is impossible since the number of such subgroups in a group of order 2^m is even.* The order of H is therefore divisible by 4 but not by 8. As all of the operators of odd order in H transform each operator of order 2 in G into itself but some of its operators of even order do not have this property, the operators of odd order generate a subgroup invariant under G, and H is the direct product of this subgroup and its cyclic subgroup of order 4.

Since the product of two non-invariant operators of order 2 in G is commutative with every operator of H, these two operators may be so chosen that their product is of order 12 and hence *the operators of order* 2 *in* G *must generate the dihedral rotation group of order* 24 *whenever four of the operators of* G *correspond to the same substitution in* T. As this invariant subgroup has only identity in common with the largest group of odd order contained in H, G includes the direct product of these two groups and its order is twice the order of this direct product.

If an operator of order 4 corresponds to the invariant substitution of T, all the operators of this divisor whose order is a power of two must be of order 4. Such an operator transforms each of the factors of the given direct product into itself. It transforms the subgroup of order 24 according to an operator of order 2, while it may be either commutative with the other factor or transform it according to an operator of order 2. In the former case G is the direct product of a group of odd order and the group of order 48 which involves just thirteen operators of order 2 and in which all the operators of order 4 are commutative with the two operators of order 3. This group of order 48 may also be defined by the fact that it contains the direct product of the quaternion group and the group of order 3, and is generated by this direct product and an operator of order 2 which transforms the operators of order 3 into their inverses and the quaternion group into a contragredient isomorphism with itself. In the latter of the two cases mentioned above G may be obtained by establishing an isomorphism between this group and a group whose order is twice an odd number.

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* *Transactions Amer. Math. Society*, vol. 6 (1905), p. 59.

When an operator of order 8 corresponds to the invariant substitution in T, all the operators of this divisor whose order is a power of two are of order 8. The remarks of the preceding paragraph apply directly to this case, with the exception that the given group of order 48 is replaced by the one which is generated by the cyclic group of order 24 and an operator which transforms each operator of this cyclic group into its eleventh power. Hence *there are two groups of order* 48 *which contain exactly* 13 *operators of order* 2, *twelve being conjugate, and transform their six four groups involving these operators according to just three distinct substitutions.* All the other possible groups which have this property can be obtained by dimidiating one of these two groups and a group whose order is twice an odd number and all such dimidiations give rise to groups having the required property.

We shall now consider the case where these six four groups correspond to six distinct substitutions of T. As these substitutions form a complete set of conjugates and are of degree less than six, T is either the positive group of order 24 or the group of order 48. If it were the latter, it would contain an invariant subgroup of order 8 times the order of H. As this subgroup would contain only one operator of order 2, it would include a Sylow subgroup having this property. The order of this S could not be less than sixteen. Hence it would have operators of order 8. This is impossible. In fact it is at once evident that such an S could not have an $(\alpha, 1)$ isomorphism with a group involving no operator of order 4 unless the order of this group is less than eight. The groups in question must, therefore, transform the six conjugate four groups according to the positive group of order 24, which is simply isomorphic with the symmetric group of degree 4.

The order of H is not divisible by four, since the operators which correspond to the three conjugate substitutions in T

must be all of a higher order than those of H. This follows also from the fact that the quaternion group is the only group of order 2^m involving only one operator of order 2 which has an operator of odd order in its group of isomorphisms. Hence H is the direct product of a group of order 2 and some group of odd order. Every operator of H is commutative with each of the operators of the group generated by all the operators of order 2 in G. Two of these operators corresponding to two commutative substitutions in T generate the octic group. Hence all the operators of H are commutative with the operators of order 4 corresponding to the three conjugate substitutions of T. That is, *G contains an invariant quaternion group and this subgroup includes all its operators of order* 4.

Two operators of order 2 can be so selected that their product is of order 3. This product and the given quaternion group generate a group of order 24 which is transformed into itself by any one of the operators of order 2. Hence *the thirteen operators of order* 2 *generate a group of order* 48, *and G is the direct product of this group and a group of odd order.* Moreover, any such direct product satisfies the given conditions. The given group is completely defined by the facts that it is of order 48 and is generated by thirteen operators of order 2. The main results of this section are as follows: If a group contains just thirteen operators of order 2 of which twelve are conjugate, these operators generate either the dihedral rotation group of order 24 or the group of order 48 which may be represented as a transitive group of degree 8 and contains the quaternion group invariantly. In the latter case G is the direct product of this group of order 48 and some group of odd order. In the former case G contains one of two groups of order 48 and may be constructed by dimidiating this group and a group whose order is twice an odd number.

If a group contains exactly thirteen operators of order 2 these operators must, therefore, generate one of the following five groups: The dihedral rotation group of order 24, the dihedral rotation group of order 26; the group of order 40 which contains thirteen operators of order 2; the group of order 48 which can be represented as a transitive group of degree 8 having four but not two systems of imprimitivity; or the group of order 72 which is generated by the direct product of the dihedral rotation group of order 12, an operator of order 3, and an operator of order 2 which transforms this operator of order 3 into its inverse and the non-invariant operators of order 2 in the dihedral rotation group of order 12 into themselves multiplied by the invariant operator of order 2. In case of the groups of order 40, 48 and 72 all the possible groups are obtained by forming the direct product of these groups and some group of odd order.

STANFORD UNIVERSITY,
November, 1905.

TYPES OF SERIAL ORDER.

*The Continuum as a Type of Order: An Exposition of the Modern Theory. With an Appendix on the Transfinite Numbers.** By EDWARD V. HUNTINGTON. Cambridge, Mass., The Publication Office of Harvard University, 1905. 4to. 63 pp. Price, 50 cents.

THE *Annals of Mathematics* has for some time followed the plan of printing articles expository of subjects which are little known or not easily accessible in the English language. Reprints of these articles are then placed on sale with the double and laudable purpose of making the circulation of the article wider than it would otherwise be and of helping solve the difficult problem of financing a mathematical journal.

The plan can hardly fail to succeed if all the articles are as clear in style and as just in the balance between generality and detail as is that of Professor Huntington. In point of readableness, we are inclined to think that the only other exposition of subjects connected with the foundations of mathematics which can be compared with Huntington's is that (in French) of L. Couturat.

The principal contents of the paper are the ordinal theory of integers, rational numbers, and the continuum, together with an appendix on the transfinite numbers of Cantor. It is intended for non-mathematical readers as well as for mathematicians, and therefore presupposes very little in the way of detailed knowledge, though of course it requires for complete comprehension a considerable maturity in abstract reasoning. We have noticed only one error of any consequence. It is stated in theorem 3, section 62, that "if a series is dense, it will also be dense in itself." That this is incorrect may be shown by an example (involving Cantor's order type Ω) similar to one given on page 169, volume 5, of the *Transactions*.

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On page 184 appears an expression of opinion which is repeated in one form or another in several other places: "It is not possible to give purely ordinal definitions for the sums and products of the elements of such a series. All that we could do in this direction would be to define the sums and products of some particular dense series, say the series of the rational numbers in the usual order, by the use of some extraordinal properties peculiar to that series; then since all series of the type η are ordinarily similar, the definitions set up in the standard series could be transferred to any other series of the same type by a one-to-one correspondence. This method would be wholly inadequate, however, since the ordinal correspondence could be set up in an infinite number of ways." The reviewer is unable to see why it is incorrect to fix attention on a single correspondence and to base the definition on that one. The fact that there exist an infinitude of other correspondences giving similar definitions is a theorem which has no particular bearing on the validity of the method pursued in regard to the particular one with which we work. On the other hand, we do not wish to controvert the author's opinion that, from a didactic point of view, it is better to introduce such fundamental concepts as addition and multiplication at the outset of the theory.

In this connection it seems proper to refer to a logical distinction recently stated by Huntington. The reference to it here, however, must not suggest that Huntington allows such logical subtleties to interfere with the clearness of his elementary exposition. A mathematical science is said to be determined categorically by a set of postulates or conditions if it can be proved that any two classes of objects satisfying the conditions are capable of a one-to-one correspondence preserving the relations described in the postulates. For example, the Cantor definition of the linear continuum in terms of ordinal relations is such that any two sets of objects satisfying the conditions imposed (for example, the points of a parabola and the points of a straight line) are capable of a one-to-one correspondence preserving order. Huntington proposes to distinguish among the ways in which a set of conditions may be categorical.* In the case just mentioned the one-to-one correspondence can be set up in an infinitude of ways, whereas if the continuum is defined in terms of postulates about addition and multiplication as well as order, the correspondence can occur in only one way. This is in part because there now are specified in any set of elements satisfying the postulates two singular elements, zero and unity, which must correspond to the zero and unity elements of any other set which satisfies the postulates.

The distinction suggested by Huntington can be carried a step further by counting parameters. Thus in the case of the linear continuum, if the postulates are stated in terms of order and addition alone there is only one singular element, zero, and the unity element may be chosen arbitrarily. Thus we have ∞^1 correspondences. If the zero is also allowed to be arbitrary but a relation of equality of segments (such as the additive operation gives by the equation $b - a = b' - a'$) be retained, then we have ∞^2 correspondences. If order alone is used in the definition then Cantor's method of showing the correspondence gives ∞^∞ correspondences. That is, there exists no n such that ∞^n will describe the possibilities of choice. We seem to have arrived at a criterion by which, given two systems of axioms, one may determine which system is, as it were, the more categorical.

In plane geometry there is an interesting question which can be decided in the same way. Hilbert's Festschrift, in addition to its axioms of connection and order and its archimedean axiom, makes use of assumptions about congruence of segments and angles, a relation which it has long been known may be introduced by definition. This definition involves the choice of an arbitrary elliptic involution on the line at infinity (*i. e.*, of two conjugate imaginary points, the circular points) and thus may be made in ∞^2 ways. The question, therefore, arises whether congruence, which as an undefined symbol in the presence of order and continuity axioms is clearly redundant, does not after all make the system of axioms more categorical?

In setting up the one-to-one correspondence between any two classes satisfying the axioms of plane geometry a pair of coördinate axes for an analytic geometry is selected in each class.

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* Cf. page 159 of the paper under review and also *Transactions Amer. Math. Society*, vol. 6 (1905), p. 41.

If congruence axioms are present, a given line a of one class may correspond to any line a' of the ∞^2 lines in the other class. A given point A of a may correspond to any one of the ∞^1 points of a'. A line perpendicular to a at A, however, must correspond to a line perpendicular to a' at A'. The whole correspondence is therefore fixed when a unit of length $A'B'$ is chosen on a' to correspond to AB on a. There are thus ∞^4 ways in which the correspondence between two planes defined by order and congruence axioms may be set up. If order axioms alone are used, it is clear that the number of correspondences must be the same as the number of collineations of a cartesian plane into itself leaving the line at infinity invariant, and this count is ∞^6. A system of axioms stated in terms of order and congruence combined is therefore more categorical than one in terms of order alone.

OSWALD VEBLEN.

COLLEGE ALGEBRA.

A College Algebra. By HENRY BURCHARD FINE. Ginn and Company, 1905. viii + 595 pp.

AMONG all the text-books of college algebra, this book by Professor Fine, of Princeton, is distinguished by its broad grasp of the subject as a branch of mathematical science and by its entire freedom from the misleading statements and positive blunders which are all too common in our current text-books. What measure of success it will obtain in the class-room it would be hard to prophesy; my own impression is that on account of the close logical interdependence of its various parts the book will be found less available for those classes which wish to take up only a few detached chapters on separate topics than for those classes which can go through the whole subject in a systematic manner. However this may be, it is certainly a book which every teacher of mathematics, whether he happens to have a class in algebra or not, should own and read. In fact, if a mathematical library were to contain only one English text-book in algebra (besides the two-volume treatise of Chrystal) I should unhesitatingly recommend the present book.

The book is divided into two parts: "a preliminary part devoted to the number system of algebra, and a principal part devoted to algebra itself."

The first part (about one-seventh of the whole) contains a systematic treatment of the rational, irrational, and imaginary numbers, in the light of the researches of Dedekind and Cantor; these researches have been known on the continent for over thirty years, but are here for the first time made accessible in a satisfactory manner in an English text-book.* If there is anyone who feels that such discussions are unimportant, on the ground that the old-fashioned treatment of irrationals and imaginaries in our current algebras is well enough as it is, let him ask any freshman class in mathematics at the opening of the college year what meanings the words "irrational number" and "imaginary number" convey to them. If his experience is anything like my own, he will find that not a single member of the class has any clear-cut notion of these concepts. The replies obtained will be simply variations on the following, which are entirely typical of those I have received: "An irrational number is one which has no exact value;" "An imaginary number is one which has no value at all and does not really exist." Interesting objects, indeed! And yet these are the mystical entities which the school boy is taught to add and subtract, to multiply and divide, as if they were ordinary numbers in the arithmetical sense. No wonder that an occasional bright pupil will protest that "you cannot extract the square root of a number which hasn't got any square root," and assure you that "to put a radical sign around a minus quantity always seemed like trying to squeeze money out of a man who didn't have any"! Such are the fruits of the present half-hearted treatment of these subjects in the ordinary text-books; any fresh and rigorous discussion, like that of Professor Fine's, should therefore be cordially welcomed by every lover of common sense and common honesty in teaching.

How much of this preliminary part will be available for the class-room, every teacher must decide for himself; the college student at any rate should be made to grasp at least the main principles which underlie the work, namely: (1) that the number system is merely a system of signs which we invent for our own convenience in handling certain practical problems (like the problems of counting and measuring); (2) that the "operations" on these numbers which are almost inevitably suggested by the exigencies of those problems are essentially defined by the fundamental laws which govern them (*e. g.*, the commutative, associative, and distributive laws for addition and multiplication); and (3) that all the propositions of algebra are logically deducible from these fundamental laws without further reference to the nature of the numbers involved.

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*Since this book was published, another excellent account of the Dedekind-Cantor theory has appeared in Pierpont's Theory of functions of real variables.

It will be seen that from this point of view the definitions of the numbers themselves are not important; the fundamental laws of operation are all that we need to know. In other words, *any* system which obeys the laws of rational algebra is a system of rational numbers; *any* system which obeys the laws of real algebra is a system of real numbers; etc.* To fix our ideas, however, and also to show that the laws are not inconsistent, it is desirable to exhibit some concrete example of such a system. The principal example which Professor Fine selects for this purpose (to speak only of the real numbers) is the one proposed by Dedekind and Cantor, as already mentioned; in this system, the rational numbers are pictured as pairs of integers and the real numbers as "separations" in the series of rationals (or as "regular sequences" of rationals). Later on another example is introduced, paralleling the first, in which the real numbers are pictured as points on a line (or better, as distances along a line). For the purposes of an elementary course it would have been preferable, I think, if the author had given the principal place to the second or more geometrical of these two parallel methods of representation. In all the formal processes of real algebra, it makes no difference whether the letters a, b, x, y, etc., stand for "separations in a series of pairs of integers" or for "distances along a line"; and surely the latter interpretation is more accessible to an elementary student, and much more useful in practice. The only advantage which the Dedekind-Cantor example possesses lies in the fact that it enables us to establish the consistency of the laws of real algebra on purely logical grounds, without any appeal to geometrical intuitions — considerations which may properly be postponed until a later stage in the student's mathematical study.

Turning now to the main part of the book, on algebra proper, we find a full discussion of "everything relating to algebra that a student is likely to need during his school and college course.'' The only topic which one will seek in vain is the binary scale of notation, which the author would regard, I suppose, as belonging more properly in arithmetic. The treatment of the various topics is fresh and stimulating, and is well calculated to keep before the student, in their proper relations, the two essentials of a mastery of the subject, namely sound reasoning and skillful manipulation.

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* Cf. *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 209–229, especially § 10.

It is of course true that no human reasoning can be more than approximately rigorous, and that in elementary work it is not desirable to carry this approximation beyond the limits of understanding of the pupil; but surely some advance over our current text-books is to be sought. A good example of what seems to me a proper degree of rigor in an elementary text-book is the excellent discussion of the solution of equations on page 114: "It is important to remember that the mere fact that a certain value of x has been derived from an equation by the rules of reckoning does not prove it to be a root; the process must be *reversible* to warrant this conclusion." This is not a mere subtlety inserted for the sake of an appearance of rigor; it is a bit of common sense which is absolutely indispensable even if we wish merely to "get the right answer." For example (page 288), in solving the simple irrational equation $x - 7 - \sqrt{x - 5} = 0$, we find that *if* x is a root, *then* x is 9 or 6; but it does not follow that (conversely) *if* x is 9 or 6, *then* x is a root; as a matter of fact, the value 6 is not a root. There is no more reason for supposing that the converse of a true proposition is also true in algebra than there is for making the same mistake in geometry. It is to be regretted that the exercises on page 290 were not so chosen as to emphasize this point.

The early introduction of synthetic division and of the method of undetermined coefficients is to be commended, as is also the use of graphs in connection with equations of the first and second degrees; in this graphical discussion, however, it might have been well to point out the sharp distinction between a *curve* as the picture of a function of one variable and a *surface* as the picture of a function of two variables.

The chapter on simultaneous quadratics, while rather difficult, will well repay a careful study, since it replaces the usual list of special devices by a connected treatment of the subject as a logical whole. It should be noticed, however, that equations like $x^2 + 3y^2 = 31$, $7x^2 - 2y^2 = 10$ (page 324) should be regarded not as quadratic equations in x and y but as linear equations in x^2 and y^2; and the use of infinite solutions is of doubtful value.

In the chapter on the theory of equations it would have been well to give a few examples of numerical equations with irrational coefficients, since the equations which come up in practice are frequently of this type. Again, in the chapter on logarithms, the awkward double-headed rule for finding the characteristic which is now in common use might well have been replaced by the older and simpler rule which refers to the units' place instead of to the decimal point.

The later chapters of the book are a veritable store-house of accurate and well-selected information concerning infinite series, infinite products, continued fractions, etc. Special attention should be called to the last chapter of all, which contains a brief but admirable discussion of the continuity of functions of one and two variables, including correct definitions of such terms as upper and lower limit and oscillation of a function, a proof of Weierstrass's theorem that a continuous function actually reaches both its upper and its lower limit in any closed interval, and Argand's proof of the fundamental theorem of algebra. This chapter is of course not intended for a first-year class, but will be valuable for reference in the later years of the student's course.

It may be noted in conclusion that the book has an excellent index, and that answers to the exercises will be supplied by the publishers on the request of a teacher.

E. V. HUNTINGTON.

FREUND'S TRANSLATION OF BALL'S HISTORY OF MATHEMATICS.

Histoire des Mathématiques. Par W. W. ROUSE BALL. Edition française revue et augmentée. Traduite sur la troisième édition anglaise par L. FREUND. Tome premier. Paris, A. Hermann, 1906. viii + 422 pp.

IT is an interesting fact that France, which produced in Paul Tannery one of the greatest, if not the greatest historian of mathematics of our time, has not for a hundred years produced a great general history of the subject. Libri was a scholarly collector of material, and a charming writer, but he did not produce a balanced history. Chasles was an investigator of unusual merit, but he never wrote a general history. Marie, with his twelve volumes based on forty years of study, produced only a mass of inaccuracies. Hoefer's little book, although mentioned as an authority by Ball, is unworthy of any attention from considerations either of style or of scholarship. Boyer's works have no merit, and Rebière's are simply amusing. Indeed France seems to have exhausted her powers in Montucla, whose work even yet is a model in style and is not without value as an authority a century and a half after the first edition appeared.

In view of these facts it is gratifying that the translator can in a measure supply what French scholars fail to produce. Zeuthen has recently been published in Paris, the Encyklopädie (a history in the best sense) is also appearing, and various other contributions to history are finding their way into the French language. It is a matter of congratulation that Ball's "Short Account of the History of Mathematics" should be included in this list. For whatever may be said against the work, the fact remains that it is one of the best arranged and most readable handbooks on the subject ever published. It has its weak features, and these have already been mentioned in various journals as the successive editions of the original have appeared, but it is doubtful if any similar work is more often used. Mr. Ball, for example, gives only about sixteen pages to his chapter on the development of arithmetic in the important period from 1300 to 1637, and less than a dozen pages to the primitive arithmetic, and he rarely mentions the subject elsewhere; he makes no mention of the Nana Ghat inscriptions; he shows familiarity with a relatively small range of historical literature, and he derives very little information from the original sources. We can reasonably pardon him for saying that the Quadrivium of Psellus appeared in 1556, although there was an edition published in Venice in 1532, and several other editions appeared before the given date. We can overlook the fact that he gives V. M. de Kempten's Practique as appearing in 1556 instead of 1550; that (page 142) he leads us to infer that the works of Cassiodorus did not appear until 1729, although the Paris edition of 1598 is well known, not to mention numerous other editions of the Compendium; that he gives 1496 as the date of publication of Sacrobosco's Algorismus, although it appeared in 1488, and that he gives 1502 instead of 1495 for the first publication of Bradwardin's arithmetic; that he asserts so positively that Campanus was a canon at Paris, and that he tells us that Ahmes was a priest, when he was only a scribe. For a not particularly scientific handbook these and similar errors may, perhaps, be overlooked for the pleasure of having at hand such a readable manual. Mr. Ball has a good eye for the selection of important topics and of interesting facts, and this has enabled him to produce a book well worth translating.

The translator has arranged to produce the work in two volumes, with a larger and more open page than the original, and the first of these volumes is the one now under review. The opportunity which came to him was an excellent one for a scholar. A real student of the history of mathematics would have made use of all of the good features of the original, would have corrected the manifest errors, and would have added bibliographical notes of value and amplified the text where it was necessary. Unfortunately Lieutenant Freund has done nothing of the sort. He has merely made a mediocre translation, adding a few footnotes of little value, and showing that his taste is really in the line of physics by an appendix devoted almost exclusively to this subject and to logarithms. His translation is not free from errors, it is carelessly made even when not otherwise objectionable, and it preserves all of the blemishes of the original.

That the work is carelessly done is seen in such features as the common lack of uniformity in titles. Ball's Mathematical Recreations is sometimes cited in French, and sometimes in English, and with the dates 1899 (page 21), 1898 (page 228), each of which should be 1897 according to the advertisement on the cover. (See also pages 31, 39, 128). Paul Tannery appears sometimes as P. and sometimes as S. P. Tannery, although always in the latter form in the original. On page 192 the title of Taylor's work is given differently in the two notes. On page 190, "De Karlsruhe" should not be in small capitals, nor should "Zwickau" on page 222. The fact that a list of names occupying seven lines on page 49 is omitted, for no apparent reason, renders line eight meaningless: "Parmi tous les mathématiciens dont nous venons de donner les noms." In the matter of proper names the treatment is even less satisfactory than in the original. Althought it is gratifying to see such Greek forms as Antiphon and Heron instead of Antipho and Hero, the French forms are often used even in cases where the classical ones are the more common. The translator also takes liberties that are quite unwarranted, as when he uses sacro-Bosco for Sacrobosco. If he were to change the form used by Ball it would have been better to adopt a more common one like Sacrobusto or Sacro Bosco.

That the footnotes added by Lieutenant Freund are of little value is seen in the fact that there are only a dozen all told. Of these, four mention articles by Loria, one mentions a work by Sedillot, one devotes seven lines to giving the useless information as to who the Admirable Crichton was (and then misspells his name!), and one says that Charles Kingsley wrote a novel about Hypatia. Of recent literature on the history of mathematics there is nothing save the four references to Loria. Hoefer is retained as an authority, but Braunmühl is unknown; Marie is retained, but there is no mention of Zeuthen or of those who are making the Abhandlungen a storehouse of first-hand material.

That no originality of scholarship is shown by the translator in his work is seen not only in the paucity of his footnotes but in the appended matter. This consists merely of a series of extracts from the works of various writers as follows: 1. A brief note by Chasles on Vieta (page 327); 2. An extended note by Biot on Napier and the invention of logarithms (pages 328–353); 3. A note, chiefly from Bertrand, on Kepler (pages 354–358); 4. A note by Mach on the development of the principles of dynamics (pp. 359–409); 5. A note by Duhem on the origin of statics (pages 410–412).

That Lieutenant Freund has preserved all of the errors and doubtful statements of the original is continually manifest. For example, we do not know, as Ball asserts, that the Mexicans used the abacus (page 132); the figure on page 135 is not a Chinese swan pan but a Japanese saroban; Alkwarizmi was born in Khuwarism, not in Khorassan (page 165); we do not know that Fibonnacci was born in 1175 (page 175), nor that Psellus was born in 1020, nor is there justification for similar positive statements not infrequently made as to dates; Widman did not use the + and — "pour indiquer l'addition et la subtraction," but for a purpose explained later in the book; Recorde did not use the symbol Z to indicate equality of ratios, for, to select one of his examples, he says: "I ſet them one ower the other, and 8 vndermoſt, thus $\begin{smallmatrix}3\\8\end{smallmatrix}\!Z$, with ſuch a crooked draught of lines. Then doe I ſet the other number which is 16, againſt 3 at the right ſide of the line, thus $^{3}_{8}\diagup^{16}$" In other words, the lines merely indicate a direction of operations, not any equality of ratios.

In general the mere translation seems satisfactory. It is a question however whether Recorde had in mind any such idea as is found in the interpretation of his "Grounde of Artes" as meaning "Jardin des Artes" (page 221).

That the book abounds in typographical errors is shown by the list given below. This is merely the result of a hurried reading, and is doubtless very incomplete. It shows, however, such an utter lack of care on the part of the translator as to assure us that an opportunity has been missed. Such a list makes it seem almost hopeless to expect a worthier effort in the second volume. Page 21, for 1899 read 1897; page 33, for Ænopides read Œnopides; page 39, for Ritz Patrick read Fitz Patrick; page 49, for Ménœchme read Ménæchme, and so on page 414, and on pages 81, 82, change Menechme; page 53, for Grichen read Griechen; and for modern read modernen; page 55, for Euclides read Eucleides (De Morgan's article), and for studien read Studien (the German capitalization being elsewhere followed); page 56, for Strobacus read Strobaeus; page 81, for Grow read Gow; page 86, for F. Hiller read E. Hiller; page 87, for Chrichton read Crichton; page 90, for phœnomena read phænomena; page 93, for J. S. Greenwood read J. G. Greenwood; page 95, for *Αὔτοματα* read *Αὐτόματα*; page 101, for Hache read Hoche; page 102, for Ptolemœus read Ptolemæus, and for 3,1416 read $3{,}141\dot{6}$; page 103, for *Συναγωγη* read *Συναγωγή*; page 112, for μ^{δ} read $\mu^{\hat{o}}$ four times; page 112 for ψ read ⋔ twice; page 136, for *ἐτεως* read *ετεος*, or *ἔτεος* if the accents are taken, the inscription referring to the seventh year, not to seven years; page 143, for C. Werner read K. Werner (cf. page 145); page 147, for Malmesburg read Malmesbury; page 154, for Appolonius read Apollonius; page 155, for Batha read Bhata; page 156, for M. Kern read H. Kern; page 161, for Arhya read Arya; page 179, for algebriste read algebrista; page 183, for Medita read Inedita; page 186, for Radolt read Ratdolt; page 190, for Encyclopœdia read Encyclopædia, and drop "Supplement" from note 1; page 263, for Breitscwert read Breitschwert; page 287, for $(n - 1^{2})^{2})$ read $(n - 1)^{2}$; page 296, for $(x - x^{3})^{2}$ read $(x - x^{2})^{1}$; page 298, for Schoeten read Schooten (as on page 313, 315);

page 307, for $p\frac{1}{3}a^{\frac{1}{3}}e$ read $p^{\frac{1}{3}}a^{\frac{1}{3}}e$; page 313, for Dictionnary read Dictionary; page 316, for d read ∂ four times, and three times on page 324; page 321, insert *di* after *Bulletino*, as elsewhere.

DAVID EUGENE SMITH.

SHORTER NOTICES.

Abhandlungen zur Geschichte der mathematischen Wissenschaften. 18. Heft. Inhalt; J. L. HEIBERG, Mathematisches zu Aristotles; C. H. MÜLLER, Studien zur Geschichte der Mathematik insbesondere des mathematischen Unterrichts an der Universität Göttingen im 18. Jahrhundert; R. LINDT, Das Prinzip der virtuellen Geschwindigkeiten. Leipzig, B. G. Teubner, 1904. 196 pp. Price, 6 marks.

FROM the standpoint of the history of pure mathematics the first two parts of this volume of the *Abhandlungen* are of great interest and value. The death of Paul Tannery left no one so well prepared to speak with authority upon a question involving both Greek mathematics and literature as Professor Heiberg. Although primarily a student of the classics, this prolific scholar has so long devoted his attention to the ancient mathematicians that he has become one of the great authorities upon their contributions.

The various histories of mathematics have always recognized the impetus given to mathematics by both Plato and Aristotle, by the former in fixing the foundations, and by the latter with reference to the history and the applications of the science. There is, however, a lack of definite information regarding the mathematical contributions of both of these leaders of philosophic thought. If we try to find exactly what Plato contributed to the advance of mathematics, Cantor, Gow, Zeuthen, and even Tannery give answers that are far from satisfactory. For Aristotle this has also been the case, and hence this concontribution of Professor Heiberg is timely and welcome, especially as it throws much light on the work of Plato as well.

The essay opens with a discussion of the sources of information and then by detailed references to the works of Aristotle it shows his influence upon the subsequent work of the Greeks. In particular the influence of this writer upon the Greek mathematical terminology is shown to be much greater than would be suspected from a reading of the standard histories. Even more interesting is the study of various propositions of Euclid in relation to the writings of Aristotle, since no serious effort has been made to trace definitely the origin of the individual theorems of the Elements. Professor Heiberg also calls attention to the fact that certain propositions of Aristotle were not used by the Alexandrian master, including the one relating to the exterior angle sum of a polygon.

We are so apt to think of Göttingen as a mathematical center of the nineteenth century alone, owing largely to the influence of Gauss, that the article by Dr. Müller on mathematics in that university in the eighteenth century will dispel what has come to be a common illusion. The essay was undertaken with the approval of Professor Klein, and to him and his associates the author acknowledges his indebtedness.

Dr. Müller first discusses the general historical problem in mathematics and then devotes a chapter to Halle and Göttingen as the universities of rationalism, and to the mathematics of this great educational period. In this chapter he discusses the scientific work of the universities of the sixteenth and seventeenth centuries in general, and of the early days of Halle and Göttingen in particular. Chapter II considers the mathematics of the period of rationalism at Göttingen, and particularly the work of Segner in pure science and of Penther in the applied field. The third chapter is devoted to the mathematics of the period of Enlightenment (Aufklärung) and chiefly to the work of Kästner and Meister. The work closes with the rise of the new humanism at Göttingen, considering particularly the later work of Kästner and his followers.

The early university courses in mathematics, and the general attitude of the academic world towards this science, are among the most interesting features of the essay. Von Rohr's exposition of Wolf's summary, for example, is very suggestive of the position of mathematics in the eighteenth century. For most Americans, the fact that Kästner (1749) thought a work of one of our countrymen * worthy of translation, is a matter of no little surprise and interest.

DAVID EUGENE SMITH.

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* An exposition of the first causes of action in matter, and of the cause of gravitation, by Cadwallader Colden, N. Y., 1745.

Elemente der Vector-Analysis. Von A. H. BUCHERER. 2te Auflage, Leipzig, B. G. Teubner, 1905. viii + 103 pp.

SINCE the first edition of this book, two years ago, there have appeared several books on vector analysis and yet a second edition of this one is called for. This is an indication of the rapidly spreading use of vectorial methods in physics and a testimony to the completeness and compactness of Bucherer's little book. As the first edition was recently reviewed in the BULLETIN,* it will merely be necessary to point out the changes in the new edition. The Heaviside-Föppl notation has been abandoned for that of Lorentz and Abraham — an evident effect of the usage of the Encyklopädie. A few sections have been inserted here and there — one on division of vectors, another on the motion of an electron; but generally speaking the increase of twelve pages is due to minor alterations. As the author is very anxious to keep his book within small bounds he still resists the temptation to enter on the study of linear vector functions. So many authors do this same thing that the use of linear functions in the theory of elasticity and light, where they are a great convenience, is seriously hampered.

E. B. WILSON.

Annuaire du Bureau des Longitudes pour l'An 1906. Paris, Gauthier-Villars.

THERE have been no changes in the *Annuaire* this year to call for special remark. The notices are all devoted to eclipse literature, with special reference to the phenomenon which occurred last August. M. G. Bigourdan, however, goes into much detail on the question of the different kinds of observations which can be made during the few minutes duration of a total eclipse of the sun. To professional astronomers this summary will be found useful as containing in a handy form facts and ideas with which they are familiar, and to amateurs it will be of equal value as showing and explaining easily the many observations which it is possible for them to undertake and carry through with success. In a second article the same writer gives a brief account of the numerous expeditions which were made last summer to observe the eclipse. M. Janssen contributes an article on his own share of this work in Spain.

In reading through these articles in the *Annuaire* during the last ten years, the reviewer has been frequently struck with the lucidity and ease shown by the writers in explaining even the most technical parts of mathematical and physical problems. Is it impossible to do this in the English language? And if not, why are such summaries so rarely seen? Or, if published, why do they seem to be heavy and unattractive? Surely it cannot be the fault of the language when we have before our eyes such a master of scientific style as Huxley. Perhaps there is something to be learned from France amongst the methods which she uses in teaching her sons to write their mother tongue. ERNEST W. BROWN.

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* Volume 10, February, 1904, pp. 263–267.

Astronomical and Historical Chronology. By W. L. JORDAN. London, Longmans, Green & Co. 8vo. 70 pp.

THE object of this little book will be sufficiently gathered from the author's statement on page 9: "My argument shows that through a misunderstanding on the part of comparatively modern historians they treated as 1 B. C. the year which, when the era was first established, was called 1 A. D. by those who used ordinal, and the year 0 by those who used cardinal numbers; and that the manner in which the centuries are considered to be divided is therefore erroneous." The question is discussed historically and much space is given to an examination of the authorities. Mr. Jordan comes to the conclusion that if the year 0 be inserted, January 1, 1900 is the beginning of the new century as decreed by the English Prayer Book and the German emperor; but that with the 'vulgar' chronology, which makes 1 B. C. immediately precede 1 A. D., the new century began a year later. The author's physical ideas seem somewhat vague: he alludes (page 33) "to the absence of any common measure between days and years as being due to the fact that the motions which they respectively measure are due to the action of two independent forces — the sun's and the earth's revolving force — etc." But perhaps this is unfair: he has written an essay on the action of astral gravitation in natural phenomena. ERNEST W. BROWN.

NOTES.

AT the meeting of the London mathematical society held on January 11 the following papers were read: By Mr. J. W. NICHOLSON "On the diffraction of sound by large cylinders"; by

Dr. H. F. BAKER, "On the monogeneity of an algebraic function"; by Mr. M. J. BRILL, "On the expression of the so-called biquaternions and triquaternions with the aid of quaternary matrices"; by Dr. E. W. HOBSON, "On the representation of functions of real variables."

THE second meeting of the Missouri society of teachers of mathematics was held at Jefferson City in conjunction with the state teachers' association, December 27 and 28. Two afternoon sessions were held, at which the following papers were presented: "Maxima and minima," by G. R. DEAN; "Laboratory methods in algebra teaching," by O. E. GLENN; "The treatment of limits in elementary geometry," by A. M. WILSON. The second session was devoted to a general discussion of the question: "What should be taught in arithmetic"? Arrangements were made to enlarge the scope of the society so as to include the natural science teachers of the state.

THE association of the teachers of mathematics of the middle schools of Switzerland met at Zürich, December 9, under the presidency of Dr. E. GUBLER. The following papers were read: "Instruction in descriptive geometry," by C. EGLI; "Tendencies in the instruction of elementary geometry," by H. FEHR; "Advantages of the decimal division of angles with logarithms to four places," by M. OTTI. Professor H. FEHR was re-elected president. The next meeting will be held in October, 1906.

THE next meeting of the French association for the advancement of science will be held at Lyons, beginning August 2, under the presidency of Professor M. Lippmann, of the University of Paris.

DURING the year 1905 doctorates in mathematics were conferred by the University of Paris on the following candidates (the title of the dissertation is appended in each case): L. ZORETTI, On uniform analytic functions which possess a perfect discontinuous set of singular points; P. STOENESCO, On the propagation and extinction of plane waves in a homogeneous and translucent medium having a plane of symmetry; D. POMPEIU, On the continuity of functions of a complex variable; R. B. DE M. DE BALLORE, On continuous algebraic fractions; A. HUSSON, Application of algebraic integrals in the movement of a heavy solid around a fixed point; J. REVEILLE, Synthetic and analytic study of the displacement of a system which remains similar to itself; C. MONTEIL, Contribution to the study of currents of heat convection.

THE Paris academy of sciences announces the following prizes in mathematical sciences for 1907 and 1908. Competing memoirs should be in the hands of the secretary before the end of the year preceding that in which the respective prizes are awarded. For further details regarding those to be awarded next December, see BULLETIN, volume 11, page 337. The Poncelet prize of 2000 francs will be awarded in 1908 in pure mathematics; Grand prize of 3000 francs in 1908 for important progress in the study of the deformation of quadric surfaces; Montyon prize of 700 francs (annual) for discoveries and inventions in mechanics; Poncelet prize of 2000 francs in 1907 in applied mathematics; Fourneyron prize of 1000 francs will be awarded in 1908 for the most important advances in the theory and use of the steam turbine; Vaillant prize of 4000 francs in 1908 for perfecting the application of the principles of dynamics of fluids to the theory of the helix; Lalande prize of 540 francs (annual) for general progress in astronomy; Guzman prize of 100,000 francs for communication with any star or any planet other than Mars. As this prize will probably not be soon awarded, the capital has been invested and the accrued interest will be given as a prize every five years for important progress in astronomy. Valz prize of 460 francs (annual) for general progress in astronomy; Pontécoulant prize of 700 francs (biennial) will be awarded in 1907 for progress in celestial mechanics; Damoiseau prize of 2000 francs (triennial) will be awarded in 1908 for the consistent theory of a planet, based upon all known observations; Hébert prize of 1000 francs (annual) for discoveries or advances in the theory of electricity; Hughes prize of 2500 franc (annual) for general physics; Gaston Planté prize of 3000 francs (biennial, open only to Frenchmen) will be awarded in 1907 for electricity; La Caze prize of 10,000 francs (biennial) will be awarded in 1907 for general physics; Kastner Boursalt prize of 2000 francs (biennial) will be awarded in 1907 for progress in the application of electricity; Binoux prize of 2000 francs (1907) for the best contribution to the history of the sciences. Besides the general prizes mentioned before, a new prize known as the Leconte prize of 50,000 francs is to be awarded in 1907 and every three years thereafter for important discoveries in mathematical or natural sciences; it is open to all competitors.

A MEMORIAL to the late Dr. George Salmon, Provost of Trinity College, Dublin, was unveiled Friday, January 5, in the national cathedral of St. Patricks, Dublin. It consists of two windows in one of the chapels, a portrait of Dr. Salmon, and a Latin inscription bearing testimony to his mathematical and theological work.

PROFESSOR P. STÄCKEL, of the technical school at Hanover, has received the order of the iron cross of the third class from the Emperor of Austria.

PROFESSOR J. THOMAE, of the University of Jena, has been decorated with the order of the cross of Saxe-Weimar by the King of Saxony.

PROFESSOR W. v. DYCK, of the technical school at Munich, has been awarded the Bavarian silver medal by Prince Luitpold.

DR. H. BRUNN has been promoted to an honorary associate professorship of mathematics at the University of Munich.

PROFESSOR L. SAALSCHÜTZ, of the University of Königsberg, celebrated his seventieth birthday December 1, 1905.

DR. G. FUBINI has been promoted to an associate professorship of higher analysis at the University of Catania.

DR. G. BISCONCINI has been appointed docent of rational mechanics at the University of Rome.

PROFESSOR A. FAVARO, of the University of Padua, will henceforth give a regular annual course of lectures on the history of mathematics.

PROFESSOR G. BAGNERA has been promoted to a full professorship of higher analysis at the University of Messina.

PROFESSOR E. BORTOLOTTI, of the University of Modena, has been appointed to a full professorship of the calculus at the same institution.

PROFESSOR SIMON NEWCOMB has been decorated with the Prussian order Pour le Mérite by the German emperor.

PROFESSOR O. S. STETSON has been promoted to an assistant professorship of mathematics in Syracuse University.

Mr. George Brett has been elected tutor in mathematics in the College of the City of New York.

Professor C. A. Van Velzer, head professor of mathematics at the University of Wisconsin, has resigned, the resignation to go into effect July 1, 1906. He expects to retire altogether from teaching.

We are informed by Professor Florian Cajori that the John Crerar Library in Chicago, has a complete set of the *Maandelykse Mathematische Liefhebbery*, 1754–1769, published in 17 volumes at Purmerende in the Netherlands. This is probably the oldest journal devoted to *elementary* mathematics. It devotes itself mainly to arithmetic and algebra, and contains several oddities in notation.

Recent second-hand catalogue: List und Francke, Talstrasse 2, Leipzig, catalogue No. 377, 1,121 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Adhémar (R. d'). Trois maîtres: Ampère, Cauchy, Hermite. (Extrait de *La Quinzaine.*) 1905. 8vo. 15 pp.

Birchby (W. N.). Euler's summation of series of reciprocal powers and related series. (*Colorado College Publications*, Vol. 11, pp. 191–208.) Colorado Springs, 1905. 8vo.

Bruns (H.). Wahrscheinlichkeitsrechnung und Kollektivmasslehre. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften.) Leipzig, Teubner, 1906. 8vo. 8 + 310 + 18 pp. Cloth. M. 8.40

Cauchy (A.). Oeuvres complètes, publiées sous la direction scientifique de l'Académie des sciences et sous les auspices de M. le ministre de l'instruction publique. 2e série. Tome I. Paris, Gauthier-Villars, 1905. 4to. 574 pp. Fr. 25.00

Doehlemann (K.). Projektive Geometrie in synthetischer Behandlung. 3te, vermehrte und verbesserte Auflage. (Sammlung Göschen, 72.) Leipzig, Göschen, 1905. 16mo. 181 pp. Cloth. M. 0.80

Guldberg (A.). Sur une classification des problèmes du calcul des variations. (*Rendiconti del Circolo Matematico di Palermo*, Tomo XXI.) 8vo. 9 pp.

Gutzmer (A.). See Vivanti (G.).

Lagrange (J. L.). Ueber die Lösung der unbestimmten Probleme 2. Grades. Uebersetzt und herausgegeben von E. Netto. Leipzig, 1904. 8vo. 131 pp. Cloth. M. 2.20

LEATHEM (J. G.). Volume and surface integrals used in physics. (Cambridge tracts in mathematics and mathematical physics, No. 1.) Cambridge University Press, 1905. 8vo. 6 + 48 pp. 2s. 6d.

LIEBMANN (H.). Notwendigkeit und Freiheit in der Mathematik. Akademische Antrittsvorlesung, gehalten in Leipzig am 25. Februar 1902. (Sonderabdruck aus dem Jahresbericht der Deutschen Mathematiker-Vereinigung, Vol. 14.) Leipzig, Teubner, 1905. 8vo. 21 pp. M. 0.80

MÜLLER (F.). Karl Schellbach: Rückblick auf sein wissenschaftliches Leben. Nebst zwei Schriften aus seinem Nachlass und Briefen von Jacobi, Joachimsthal und Weierstrass. Mit einem Bildnis Karl Schellbachs. (Abhandlungen zur Geschichte der mathematichen Wissenschaften mit Einschluss ihrer Anwendungen, begründet von Moritz Cantor, Heft XX, 1.) Leipzig, Teubner, 1905. 8vo.

NETTO (E.). See LAGRANGE (J. L.).

NIELSEN (N.). Handbuch der Theorie der Gammafunktion. Leipzig, Teubner, 1906. 8vo. 10 + 326 pp. Cloth. M. 12.00

PICARD (E.). La science moderne et son état. (Bibliothèque de philosophie scientifique.) Paris, 1905. 12mo. Fr. 3.50

SCHEFFERS (G.). See SERRET-HARNACK.

SCHUBERT (H.). Auslese aus meiner Unterrichts- und Vorlesungspraxis. Vol. II. Leipzig, Göschen, 1905. 8vo. 218 pp. Cloth. M. 4.00

SERRET-HARNACK. Lehrbuch der Differential- und Integralrechnung. (In 3 Bänden.) Vol. I. 3te Auflage, besorgt von G. Scheffers. Leipzig, Teubner, 1906. 8vo. 600 pp. Cloth.

VIVANTI (G.). Theorie der eindeutigen analytischen Funktionen. Unter Mitwirkung des Verfassers deutsch herausgegeben von A. Gutzmer, Leipzig, Teubner, 1905. 8vo. 300 pp. Cloth.

II. ELEMENTARY MATHEMATICS.

ALBRICH (K.). Lehrsätze aus der Geometrie. Ein Hilfsbuch für die unteren Klassen der Mittelschule. Nagyszeben, Krafft, 1905. 8vo. 36 pp. M. 0.25

——. Ziel und Methode des mathematischen Unterrichtes an der Mittelschule. 8vo. 35 pp. M. 0.42

BENÍTEZ Y PARODI (M.). See SALINAS Y ANGULO (J.).

CARTER (C.). Graphic algebra for beginners. (Normal tutorial series.) London, Simpkin, 1905. 8vo. 1s. 6d.

GIROD (J.). Précis de géométrie plane, conforme aux programmes du 27 juillet 1905, pour les classes de seconde C et D et de mathématiques A et B. Paris, Alcan, 1905. 8vo. 12 + 252 pp. Fr. 2.50

HAACKE (F.). Entwurf eines arithmetischen Lehrganges für höhere Schulen. Teil II. Wohlau, 1905. 8vo. 23 pp. M. 1.50

HADAMARD (J.). Leçons de géométrie élémentaire. Vol. I: Géométrie plane. 2e édition, revue et corrigée. Paris, Colin, 1906. 8vo. 16 + 309 pp. Fr. 6.00

HAUCK (G.). Lehrbuch der Stereometrie. Auf Grund von Dr. F. Kommerell's Lehrbuch neu bearbeitet und erweitert. 9te Auflage (8te der Neubearbeitung.) Herausgegeben von Dr. V. Kommerell. Tübingen, Laupp, 1905. 8vo. 15 + 224 pp. M. 2.60

KOMMERELL (V.). See HAUCK (G.).

Loud (F. H.). Solution of numerical cubic equations. (*Colorado College Publications*, Vol. 11, pp. 219–224.) Colorado Springs, 1905. 8vo.

Macé de Lépinay (A.). See Vacquant (C.).

Mahler (G.). Ebene Geometrie. 4te, verbesserte Auflage. (Sammlung Göschen, 41). Leipzig, Göschen, 1905. 16mo. 116 pp. Cloth. M. 0.80

Martinaud (A.). Cours d'algèbre élémentaire, à l'usage de l'enseignement secondaire, rédigé conformément aux programmes officiels du 27 juillet 1905, et renfermant les matières des programmes d'admission aux écoles d'arts et métiers et de commerce (premier cycle). Paris, Belin, 1905. 12mo. 358 pp.

Oca (E.). Nociones de geometria, con especial aplicación á la agrimensura, división, agregación y segregación de terrenos y medición de rectas inaccesibles. 7a edición. Logroño, Oca, 1905. 56 pp. Fr. 6.00

Octavio de Toledo (L.). Tratado de algebra. Tomo 1: Parte elemental. Madrid, Murillo, 1905. 359 pp. Fr. 10.00

Premiers exercices d'algèbre, à l'usage des écoles primaires. 2e édition. Lyon, Crozier, 1905. 16mo. 103 pp.

Rogel (F.). Das Rechnen mit Vorteil. Eine gemeinfassliche, durch zahlreiche Beispiele erläuterte Darstellung empfehlenswerter Vorteile und abkürzender Verfahren. Leipzig, Teubner, 1905. 8vo. 4 + 38 pp. M. 0.80

Salinas y Angulo (I.) y Benítez y Parodi (M.). Algebra. Elegida de texto en el concurso celebrado el 28 de febrero de 1885 por la Dirección general de Instrucción militar. 4a edición. Madrid, Arias, 1905. 19 + 772 pp. Fr. 18.00

Sassenfeld (J.). Hauptsätze der Elementar-Mathematik für das Gymnasium, mit Aufgaben-Sammlung in der Algebra. 2te Auflage. Unterstufe. Trier, Lintz, 1906. 8vo. 8 + 188 pp. Cloth. M. 3.20

Simon (M.). Die Entwickelung der Elementargeometrie im XIX. Jahrhundert. Referat erstattet der Deutschen Mathematiker-Vereinigung. (Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsband I.) Leipzig, Teubner, 1905. 8vo. 260 pp.

Stern (H. A.) and Topham (W. H.). Practical Mathematics. Part 2. London, Bell, 1905. 8vo. 274 pp. Cloth. 3s. 6d

Topham (W. H.). See Stern (H. A.).

Vacquant (C.) et Macé de Lépinay (A.). Géométrie élémentaire (premières notions), à l'usage des classes de lettres. Géométrie plane (classes de quatrième et de troisième A). 14e édition. Paris, Masson, 1906. 16mo. 211 pp. Fr. 2.00

——. Premiers éléments de géométrie, à l'usage des classes de sciences (premier cycle). Paris, Masson, 1906. 16mo. 439 pp. Fr. 3.50

Weber (H.) und Wellstein (J.). Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. Vol. I: Weber (H.), Encyklopädie der elementaren Algebra und Analysis. 2te Auflage. Leipzig, Teubner, 1906. 8vo. 18 + 540 pp. Cloth. M. 9.60

III. APPLIED MATHEMATICS.

Boltzmann (L.). Populäre Studien. Leipzig, 1905. 8vo. 7 + 440 pp. M. 8.00

Dessau (O.). See Righi (A.).

Felgentraeger (W.). Theorie, Konstruktion und Gebrauch der feineren Hebelwage. Leipzig, Teubner, 1905. 8vo. 320 pp. Cloth.

Ferval (H.). Eléments de géométrie descriptive, à l'usage des candidats aux baccalauréats de l'enseignement secondaire et aux écoles du gouvernement. 6e éditon. Paris, Belin, 1905. 12mo. 6 + 326 pp.

Greene (C. E.). Structural mechanics; comprising the strength and resistance of materials, and elements of structural design. With examples and problems. Revised by A. E. Greene. 2d edition. New York, 1905. 8vo. 241 pp. Cloth. $2.50

Harrison (H. H.). Engineering mathematics simply explained. Textbook for apprentices, students, and engineers. London, Marshall, 1905. 8vo. 166 pp. 1s. 6d.

Hay (A.). Alternating currents; their theory, generation and transformation. London, Harper, 1905. 8vo. 306 pp. Cloth. 6s.

Herrmann (J.). Elektrotechnik. Einführung in die moderne Gleich- und Wechselstromtechnik. Teil 3: Die Wechselstromtechnik. Kurze Darstellung der Gesetze des Wechselstromes und Beschreibung der Generatoren, Transformatoren und Motoren für Wechselstrom. (Sammlung Göschen, 198.) Leipzig, Göschen, 1905. 16mo. 140 pp. Cloth. M. 0.80

Husson (E.). Recherche des intégrales algébriques dans le mouvement d'un solide pesant autour d'un point fixe. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 85 pp.

James (W. H. N.) and Sands (D. L.). Elementary electrical calculations. A book suitable for the use of students of electrical engineering. New York, Longmans, 1905. 8vo. 216 pp. Cloth. $1.25

Landré (C. L.). Mathematisch-technische Kapitel zur Lebensversicherung. 3te, verbesserte und vermehrte Auflage. Jena, 1905. 8vo. 24 + 506 pp. M. 11.00

Lebon (E.). Géométrie descriptive et géométrie cotée, conforme aux programmes du 31 mai 1902 pour l'enseignement secondaire. Classes de mathématiques A et B. Paris, Delalain, 1905. 8vo. 6 + 176 pp. Fr. 3.50

Manouvrier (G.). Traité élémentaire de mécanique. Nouvelle édition, entièrement refondue. Paris, Hachette, 1905. 16mo. 367 pp. Fr. 4.00

Preskot (R.). Die näherungsweise Berechnung der Compensationspendel. Leipzig, 1905. 8vo. 29 pp. M. 0.75

Réveille (M. J.). Etude synthétique et analytique du déplacement d'un système qui reste semblable à lui-même. (Thèse.) Paris, Challamel, 1905. 4to. 157 pp.

Righi (A.). Die moderne Theorie der physikalischen Erscheinungen (Radioaktivität, Ionen, Elektronen). Uebersetzt von B. Dessau. Leipzig, 1905. 8vo. 5 + 172 pp. M. 2.80

Rohr (M. von). Die optischen Instrumente. Leipzig, 1906. 8vo. 130 pp. M. 1.00

Sands (D. L.). See James (W. H. N.).

Schleschka (J.). See Wildt (J.)

Wildt (J.) und Schleschka (J.). Leitfaden für den Unterricht in der Geometrie und Projectionslehre. Wien, 1905. 8vo. 90 pp. M. 2.00

Wood (R. W.). Physical optics. New York, Macmillan, 1905. 8vo. 13 + 546 pp. Cloth. $3.50

LIBRARY.

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and twenty-seventh regular meeting of the Society was held in New York City on Saturday, February 24. The attendance at the two sessions included the following thirty members of the Society:

Professor G. A. Bliss, Professor C. L. Bouton, Professor Joseph Bowden, Dr. W. H. Bussey, Professor F. N. Cole, Dr. W. S. Dennett, Professor L. P. Eisenhart, Professor H. B. Fine, Mr. A. M. Hiltebeitel, Dr. Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Professor Max Mason, Mr. A. R. Maxson, Professor H. B. Mitchell, Professor Richard Morris, Professor W. F. Osgood, Miss I. M. Schottenfels, Professor Charlotte A. Scott, Professor C. S. Slichter, Dr. Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Mr. A. W. Stamper, Dr. C. E. Stromquist, Professor H. D. Thompson, Professor E. J. Townsend, Professor Oswald Veblen, Miss E. C. Williams, Professor J. W. Young.

The President of the Society, Professor W. F. Osgood, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. M. J. Babb, University of Pennsylvania; Mr. William Betz, East High School, Rochester, N. Y.; Mr. G. D. Birkhoff, University of Chicago; Mr. W. C. Breuke, Harvard University; Mr. B. E. Carter, Massachusetts Institute of Technology; Dr. H. L. Coar, University of Illinois; Miss Anna Johnson, Harvard University; Mr. W. D. Lambert, U. S. Coast Survey; Mr. W. A. Luby, Central High School, Kansas City, Mo.; President W. J. Milne, New York State Normal College; Professor Richard Morris, Rutgers College; Mr. W. J. Newlin, Harvard University; Miss R. A. Pesta, Wendell Phillips High School, Chicago, Ill.; Dr. H. B. Phillips, University of Cincinnati; Mr. A. R. Schweitzer, University of Chicago; Mr. C. G. Simpson, Michigan College of Mines; Mr. A. W. Stamper, Columbia University; Mr. F. C. Touton, Central High School, Kansas City, Mo.; Mr. M. O. Tripp, College of the City of New York. Ten applications for admission to the Society were received.

The date of the next annual meeting of the Society was fixed as Friday–Saturday, December 28–29. The summer meeting and colloquium will be held at Yale University, extending through the week September 3–8. Courses of colloquium lectures have already been arranged, and a preliminary announcement will be issued in May.

The following papers were read at the February meeting:

(1) Dr. W. H. BUSSEY: "On the tactical problem of Steiner."

(2) Miss I. M. SCHOTTENFELS: "On linear fractional transformations of functions of the complex variable $u + \epsilon v$, where $\epsilon^2 = 0$" (preliminary communication).

(3) Professor C. J. KEYSER: "On the linear complex of circle ranges in a plane."

(4) Professor E. B. WILSON: "Note on integrating factors."

(5) Miss R. L. CARSTENS: "A set of independent postulates for quaternions."

(6) Dr. W. B. FORD: "On the analytic extension of functions defined by double power series."

(7) Professor OSWALD VEBLEN: "Remark on a measure of categoricalness."

(8) Professor VIRGIL SNYDER: "Surfaces generated by conics cutting a twisted quartic curve and a line in the plane of the conic."

(9) Dr. CLARA E. SMITH: "Development of a function in terms of Bessel's functions (second paper)."

(10) Professor L. P. EISENHART: "Surfaces with the same spherical representation of their lines of curvature as spherical surfaces."

(11) Professor PAUL STÄCKEL: "Die kinematische Erzeugung von Minimalflächen (erste Abhandlung)."

(12) Professor OSKAR BOLZA: "A fifth necessary condition for a strong extremum of the integral

$$\int_{x_0}^{x_1} F(x, y, y')dx."$$

Miss Carstens's paper was communicated to the Society by Dr. Epsteen, Professor Stäckel's by Professor E. H. Moore. In the absence of the authors, the papers of Professor Wilson, Miss Carstens, Dr. Ford, Professor Snyder, Professor Stäckel and Professor Bolza were read by title. Abstracts of the papers follow below; the abstracts are numbered to correspond to the titles in the list above.

1. The problem considered in Dr. Bussey's paper was proposed by Steiner in the *Journal für die reine und angewandte Mathematik*, volume 45, page 181. It has to do with the arrangement of N elements in triads, tetrads, pentads, etc. That part of the problem which relates to triads has been solved (see Encyclopédie des Sciences mathématiques, volume 1, page 80). The other parts are more difficult and have been made the object of but little study. By means of the properties of linear homogeneous equations in the Galois field of order 2, the author has completed the solution for the linear triple systems in $2^k - 1$ elements.

2. Miss Schottenfels treated the developments essential to the study of the linear fractional transformations of functions of the complex number $u + \epsilon v$, where $\epsilon^2 = 0$, including certain transformations such as $z' = z + \epsilon$, $z' = \mu z$.

3. Professor Keyser's paper, like that presented by him at the January meeting, deals with the circle range geometry of the plane. The range enjoys four degrees of indetermination. The ranges satisfying a single condition constitute a complex. A pencil of ranges is the ensemble of ranges having a common circle and lying in a circle congruence (totality of circles orthogonal to a given circle). The degree of a complex of ranges is the number of ranges common to it and an arbitrary pencil. When this number is 1, the complex is linear. The ranges of such a complex that have a given circle in common constitute a pencil, polar to the given circle. The ranges of the complex that lie in the congruence of a given pencil constitute a pencil, pole of the congruence. Hence a linear complex is a means of reciprocal transformation pairing the circles with the congruences of the plane and the ranges with the ranges. The ranges of a pair are conjugates. The circles of a range correspond to the congruences of the conjugate range, and reciprocally. The anharmonic ratio of any four circles of a range is equal to that of the corresponding congruences of the conjugate range, and reciprocally. To any configuration of circles, ranges and congruences corresponds a (reciprocal) configuration of (the polar) congruences, ranges and circles. Every range having a circle in common with each of two con-

jugate ranges belongs to the complex. If a range r of the complex has a circle in common with a given range r_1, r contains a circle of the conjugate range r_1' of r_1. According as two conjugate ranges contain or do not contain a common circle (or congruence) they do or do not belong to the complex. If they do, they coincide. Three independent ranges r_1, r_2, r_3 determine an infinity of ranges each intersecting each of the given r's in a circle. The ranges so determined are readily constructible. They constitute a system S of generating ranges of a quadric configuration of ranges, analogous to the simple hyperboloid of space. The second system S' of generating ranges is determined by any three ranges of S. If the ranges of S belong to a complex C, they are each self-conjugate as to C. Then no range of S is in C but the conjugate of every range of S is in S'. The radical axes of the ranges in S envelope a conic; similarly for S', and the conics coincide. In case of the analogue of the hyperbolic paraboloid, the conic degenerates into a pair of pencils of lines.

4. Professor Wilson gave an elementary proof of the relation $M\sqrt{X_1^2 + X_2^2 + \cdots + X_n^2} = dF/dn$ between the coefficients X_i, the integrating factor M, and the solution F of an integrable total differential equation $\sum X_i dx_i = 0$. This relation was then used to discuss, geometrically and somewhat more in detail than is usual, the matter of singular solutions, limiting solutions and the factors of $M^{-1} = 0$. The paper is to appear in the *Annals of Mathematics*.

5. In this paper Miss Carstens defines Hamilton's quaternions by a set of independent postulates based on Professor Dickson postulates for hypercomplex number systems (*Transactions*, volume 6, No. 3, pages 344–348; 1905).

6. Dr. Ford's paper considers the functions $f(x, y)$ defined by the double power series

$$(1) \qquad \sum_0^\infty \sum_0^\infty a(m, n) x^m y^n.$$

It is shown that under certain conditions for the function $a(m, n)$ the function $f(x, y)$ may be extended analytically outside the circles of convergence of the series (1) and throughout the entire x, y planes with the exception of the cuts 0 to $+\infty$ along the real x and y axes. The paper will appear in the *Transactions*.

7. Professor Veblen's note appeared in the March BULLETIN as the last part of his review of Huntington's Types of Serial Order.

8. If a line and a twisted quartic be given, and a correlation between the planes and points of the line be established, then a conic is uniquely fixed by the four points on the curve and the point associated with the plane. The surfaces studied by Professor Snyder are described by the conic when the plane turns about the line, which was chosen in various ways with regard to the curve.

9. Schlömilch determined the coefficients of his development of an arbitrary function $f(x)$ in terms of $J_0(x)$ by applying Abel's relation

$$\frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{f'(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\sqrt{1-\lambda^2}} = f(u) - f(0),$$

to an auxiliary function in terms of which $f(x)$ could be expressed. The justification of this method involves certain conditions on the second derivatives of $f(x)$. Miss Smith showed that this unnecessary restriction can be removed by applying Abel's relation directly to $f(x)$.

Another development in terms of $J_0(x)$, whose coefficients differ slightly in form from those of Schlömilch, can be justified under slightly more general conditions. In simple cases the two developments are identical. Analogous to this second development is one in terms of $J_1(x)$, which is much more easily justified than that which Schlömilch obtained by termwise differentiation of the series in $J_0(x)$. These also are in many cases identical.

10. In several previous papers Professor Eisenhart has considered the surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces, called for convenience A-surfaces. Now he discusses the case where the representation is that of spherical surfaces, that is, surfaces with constant positive curvature, pointing out his previous results which have a significance in the present case and estab-

lishing theorems which of necessity have no analogues in the other theory. There exists an imaginary transformation of one of the surfaces into another of the same kind, which is similar to the generalized Bäcklund transformations of A-surfaces. Pairs of these transformations can be found which when applied successively to a real surface yield a new real surface of the same kind. Bonnet showed that these surfaces go in pairs — the members of a pair being applicable with correspondence of the lines of curvature; and, moreover, these surfaces are the only ones applicable in this manner. On this account we call them surfaces of Bonnet. The knowledge of a transformation of such a surface enables one to find by algebraic processes the surface of Bonnet applicable to the given one. By means of the above-mentioned transformations a pair of real applicable surfaces of Bonnet can be transformed into a new pair. There exist a large number of surfaces whose coördinates are expressed in forms similar to those for surfaces of Bonnet and the surfaces analogous to the latter considered elsewhere by the author.

11. A curved surface is said to have a kinematic generation if it is generated by a rigid curve moving according to a given law. To surfaces generated kinematically belong the minimal surfaces whose generators are imaginary curves with vanishing line element. Professor Stäckel proposes to publish in a series of articles the results of investigations which he is making on those exceptional minimal surfaces that have more than one kinematic generation. The first paper, which will appear in the *Transactions*, is concerned with the minimal surfaces that can be determined in more than one way as translation surfaces. A new and direct proof is given of Lie's theorem that the Scherk's surfaces are the only minimal surfaces which can be generated by the translation of curves with non-vanishing line element. For, if it be required that a surface of translation be at the same time a minimal surface, then the gaussian parameter representation gives a functional equation for three functions of u and three functions of v. A complete solution for this equation is determined. If the generating curve is plane, the Scherk's surfaces are obtained immediately; but if it is a twisted curve in space, then the solution of the given functional equation reduces to the integration of the system of ordinary differential equations

$$\frac{hg' - gh'}{f^2 + g^2 + h^2} = s_{11}f + s_{12}g + s_{13}h,$$

$$\frac{fh' - hf'}{f^2 + g^2 + h^2} = s_{21}f + s_{22}g + s_{23}h,$$

$$\frac{gf' - fg'}{f^2 + g^2 + h^2} = s_{31}f + s_{32}g + s_{33}h,$$

in which f, g, h are functions of u and f', g', h' their derivatives, while $s_{11}, \cdots, s_{33}$ are constants. These equations also give the Scherk's surfaces, which admit an infinite number of generations in the way described; moreover the ordinary helicoidal surfaces appear as special cases of this result.

12. It is well known that the conditions of Euler, Legendre, Jacobi, and Weierstrass are not sufficient for a strong extremum of the integral

$$J = \int_{x_0}^{x_1} F(x, y, y')dx.$$

In Professor Bolza's paper a fifth necessary condition is established. The paper will be published in the current volume of the *Transactions*.

F. N. COLE,
Secretary.

THE FIFTY-FIFTH ANNUAL MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE American Association for the Advancement of Science held its fifty-fifth annual meeting in New Orleans, the sessions continuing from December 28, 1905, to January 3, 1906.

The president of the meeting was Professor Calvin M. Woodward of St. Louis. Dr. L. O. Howard, Washington, D. C., is the permanent secretary of the Association. The enrollment was small, reaching a total of only 233, and the programmes of many of the sections were unusually brief, but the meeting as a whole can by no means be considered unsuccessful. It is believed by many that, though the attendance may always be small, one of the most important purposes of the organization — the stimulation of popular interest in science — will be better achieved by meeting more frequently in the remoter cities of the country. It is to be noted, however, that few of the affiliated societies find it advisable to meet at a great distance from the more populous educational centers. Thus, another important function of the Association — the coördination of scientific interests — is certain to be suspended as often as meetings at distant points are undertaken. It should in any case be said, with reference to the New Orleans meeting, that the welcome accorded the Association was remarkably cordial.

The address of the retiring president, Professor W. G. Farlow, of Cambridge, Mass., was given at Sophie Newcomb College on the evening of December 29, the subject being 'The popular conception of the scientific man at the present day.' It has been published in full in *Science* for January 5 of the current year.

The meetings of Section A (mathematics and astronomy), as well as those of the other sections, were held in the buildings of Tulane University. The officers of the section were: vice-president, W. S. Eichelberger; secretary, L. G. Weld; councilor, C. S. Howe; member of the general committee, G. B. Halsted; press secretary, the secretary of the section; sectional committee, Alexander Ziwet, J. R. Eastman, Ormond Stone, E. B. Frost, E. O. Lovett, Harris Hancock, together with the vice-president and the secretary of the section. In the absence of the vice-president the retiring vice-president, Professor Ziwet, acted as chairman. The following mathematicians and astronomers were, upon nomination by the sectional committee, elected by the council to fellowship in the Association: E. W. Brown, W. A. Granville, Harris Hancock, M. W. Haskell, E. J. Townsend, Irving Stringham, Paul Wernicke.

The next annual meeting of the Association will be convened in New York City, on Thursday, December 27, 1906. In addition to this meeting it was decided by the general committee to hold a special summer meeting at Ithaca, New York, to close on or before July 3. There will be no presidential or vice-presidential addresses at the summer meeting. All officers elected at the New Orleans meeting will hold over to the close of the New York meeting. Dr. W. H. Welch of Baltimore, Md., was elected president of the New York and Ithaca meetings, Dr. Edward Kasner of New York City will be vice-president of section A. The present secretary of the section is to continue in office. Chicago was recommended as the place of meeting in 1907.

The address of the retiring vice-president of section A, Professor Alexander Ziwet, on 'The relation of mechanics to physics,' was presented on the afternoon of December 29. It has already been published in full in *Science* for January 12. At the regular programme meeting of the section, held on December 30, eleven papers were presented. The titles of these are given below, with abstracts of such as deal with purely mathematical subjects.

(1) Dr. O. E. **Glenn**: "On the groups of order $p^m q^n$ having abelian subgroups $H_{p^{mn}}$ of type $[n, n, \cdots, n]$."

(2) Professor G. B. **Halsted**: "A new straight in non-euclidean geometry."

(3) Professor **Harris Hancock**: "A chapter in the present state of development of the elliptic functions."

(4) Mr. H. B. **Hedrick**: "A catalogue of 1607 zodiacal stars for the epochs 1900 and 1920, reduced to an absolute system."

(5) Dr. **Edward Kasner**: "A class of central forces."

(6) Professor F. H. **Loud**: "Solar photographs."

(7) Professor G. A. **Miller**: "The groups of order p^m which contain exactly p cyclic subgroups of order p^a."

(8) Mr. J. J. **Quinn**: "Inversion and inversors."

(9) Professor **David Todd**: "Observations of the total solar eclipse of 1905, August 30, at Tripoli, Barbary."

(10) Professor **David Todd** and Mr. R. A. **Baker**: "Computed traces and totality durations of the total eclipses of the twentieth century."

(11) Professor L. G. **Weld**: "A possible extension of the theory of envelopes."

1. This paper is supplementary to one presented by the author at the Philadelphia meeting of the Association, in which the case $n = 1$ was discussed. The defining relations of all groups described in the title are tabulated, and their properties discussed in relation to the properties of the Galois field $GF[p^{mn}]$ determined by the automorph of the subgroup H. It is found that all of the groups in question are members of a general family and have one general set of defining relations.

2. The paper sets forth the discovery that, in Riemann non-euclidean geometry, the six mid-points of the parts of the

six rays from the vertices of any triangle obtained by prolonging the sides are costraight. This is a new and noteworthy straight associated with every triangle. The theorem is demonstrated, and then interpreted in ordinary euclidean space.

3. This paper is an attempt to show that practically all (American and European) writers on the elliptic functions have been giving too much emphasis to certain parts of Weierstrass's theory, while they have neglected many of the lines of thought which Weierstrass himself considered fundamental.

It is shown that the so-called Weierstrass normal form is not due to Weierstrass. The introduction of new functions gives a different aspect to the presentation of the elliptic functions although little that is new has been added thereby to the theory itself. Weierstrass's great work lies in a somewhat different direction. With him the problem of determining all analytic functions which have algebraic addition theorems is the leading idea.

The paper also cites several fundamental theorems of Hermite and indicates some of the characteristics of Riemann's theory.

4. This paper will be published in the Astronomical Papers of the American Ephemeris, volume 8, part 3.

5. There exists no field of force in which a particle started from an arbitrary position with arbitrary velocity will describe a circular path. In the case of a central force the only possible circular trajectories are, in general, those whose centers are at the origin of force. If, however, the force varies according to a function of the form $br(r^2 - a)^{-3}$, then a quadruple infinity of the trajectories are circular. In the simplest case, arising when a vanishes, the force varies inversely as the fifth power of the distance, and the circles all pass through the origin. In the general case they are orthogonal or diametral to a fixed sphere.

7. The main theorems proved in this paper may be stated as follows: If a group of order p^m, p being any odd prime, contains exactly p cyclic subgroups of order p^α, $\alpha > 2$, it contains exactly p cyclic subgroups of every order which exceeds p and divides p^{m-1}. Hence it is one of the two non-cyclic groups of order p^m which contain operators of order p^{m-1}. When $\alpha = 2$

and $p > 3$ the theorem is still true. In fact, the only possible exception occurs when $\alpha = 2$, $p = 3$ and $m = 4$. In this special case there are three groups which contain exactly p cyclic subgroups of order p^α.

When $p = 2$ the preceding theorem is replaced by the following: If a group of order 2^m contains exactly two cyclic subgroups of order 2^α, $\alpha > 2$, it cannot contain more than two cyclic subgroups of any higher order. If a group of order 2^m contains exactly two cyclic subgroups of order 2^β but does not contain any cyclic subgroup of order $2^{\beta+1}$, then m cannot exceed $2^{\beta+2}$. These theorems involve the fundamental properties of all possible groups whose order is a power of any prime p and which involve exactly p cyclic subgroups of any given order p^α. From a well known theorem it follows that α is not unity, but it can have every other possible value less than m.

8. In this paper are presented two new theorems relating to inversion, besides an explanation of the construction of certain linkages exhibiting the operation of inversion.

11. (*a*) In the equation $f(\alpha, x, y) = 0$, representing a family of loci, by giving to α, first an increment and then a corresponding decrement, each of magnitude $\Delta\alpha$, solving the resulting equations for the coördinates of the point of intersection and, finally, letting $\Delta\alpha \doteq 0$, there will be obtained $x' = \phi(\alpha)$, $y' = \psi(\alpha)$. These equations define a point of the envelope of the given family of loci and eliminating α between them gives $F(x', y') = 0$, the equation of the envelope.

The point (x', y'), determined as above, may be called the tracing point of the locus; that is, the point which, for the moment, is tracing the envelope. It was shown in the paper, by way of illustration, that the tracing point for the envelope of the family of ellipses

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \quad \alpha + \beta = c$$

is the Fagnani point.

(*b*) The inverse of the above notion was next developed with reference to the right line, viz: A point on the line

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1$$

being assigned at will, to find the functional relation between the intercepts, $\Phi(\alpha, \beta) = 0$ (*i. e.*, the law governing the motion of the line), in order that the given point may trace an envelope and, finally, to obtain the equation of the envelope. The required relation is given by either of the differential equations

$$x' = \phi(\alpha, \beta) = \frac{\alpha^2}{(\alpha - \beta) d\alpha / d\beta}, \quad y' = \psi(\alpha, \beta) = \frac{\beta^2}{(\beta - \alpha) d\beta / d\alpha}.$$

In general both equations will be needed in order to determine the constants of integration. Having thus obtained the function Φ, which is, in effect, the tangential equation of the envelope, the equation in rectangular coordinates readily follows.

Several examples applying the principles were presented and its application to other families of loci was suggested as a promising field of investigation for the amateur mathematician.

LAENAS GIFFORD WELD,
Secretary.

THE STATE UNIVERSITY OF IOWA,
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A PROOF OF THE FUNDAMENTAL THEOREM OF ANALYSIS SITUS.

BY PROFESSOR G. A. BLISS.

(Read before the American Mathematical Society, December 28, 1905.)

THE theorem that a Jordan curve divides the plane into two regions, an interior and an exterior, has in recent years received much attention. The proofs which have been given may be roughly divided into two classes, those in which the object has been to prove the theorem with the fewest possible hypotheses on the curve,* and those in which generality has to a certain extent been sacrificed for simplicity.† The following proof belongs to the second class. In § 1 it is assumed that the curve considered is continuous and has a continuously turning tangent at every point; but in § 3, by extending the proof of one of the auxiliary theorems, curves with a finite number of corner points and singularities are also included. The proof is presented here because it seems shorter and simpler than those heretofore given.

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* Veblen, *Transactions Amer. Math. Society*, vol. 6 (1905), p. 83.

† Ames, *Amer. Jour. of Math.*, vol. 27 (1905), p. 353. Bliss, BULLETIN, vol. 10 (1904), p. 398. For further references, see the paper by Ames.

§ 1. *Hypotheses on the Curve.*

The curve to be considered is for the present supposed to be closed, but otherwise non-intersecting, continuous with a continuously turning tangent at every point, and to have no singular points. If its equations are given in the form

$$C: \qquad x = \phi(t), \quad y = \psi(t),$$

these conditions may be expressed analytically as follows:

(*a*) The functions ϕ and ψ are periodic with period ω, but the points (x, y) defined by two different parameter values t, t' are distinct unless t and t' differ by a multiple of ω.

(*b*) ϕ and ψ are continuous for all values of t.

(*c*) The derivatives ϕ' and ψ' are continuous for all values of t.

(*d*) $\phi'^2 + \psi'^2 > 0$ for all values of t.

Let (ξ, η) be a point of the curve C defined by a parameter value τ. On account of (*d*) and (*c*) one of the derivatives, say ϕ', is different from zero in an interval $[\tau - \delta, \tau + \delta]$, and as t traverses the interval x varies monotonically between two values ξ_1 and ξ_2. Under these circumstances t is similarly a monotonic continuous function of x in the interval $[\xi_1, \xi_2]$, and this function substituted for t in $y = \phi(t)$ gives an equation in the form $y = f(x)$.

For any point (ξ, η) of the curve C there exists an interval $[\xi_1, \xi_2]$ including the value ξ, such that all the points of C near (ξ, η) satisfy an equation

$$(1) \qquad y = f(x), \quad (\xi_1 \leqq x \leqq \xi_2);$$

or else there exists an interval $[\eta_1, \eta_2]$ including η, such that

$$x = g(y), \quad (\eta_1 \leqq y \leqq \eta_2).$$

The function f, or g, is single-valued and continuous.

With the help of the last statements the following important auxiliary theorem can be readily proved:

AUXILIARY THEOREM I. *At any point (ξ, η) of the curve C, a rectangle with its center at (ξ, η) and sides parallel to the axes* *can be constructed, in such a way that it is intersected only twice and divided into two continua by the curve.*

To fix the ideas consider a point near which the curve has the form (1). A square of side 2ϵ with its center at (ξ, η) will be intersected only by the arc (1) if the half-diagonal $\epsilon\sqrt{2}$ is taken less than the shortest distance from (ξ, η) to other parts of the curve. If another positive constant δ is now taken so that in the interval $[\xi - \delta, \xi + \delta]$ the absolute value of $f(x) - \eta$ is less than ϵ, then the rectangle whose sides are $x = \xi \pm \delta$, $y = \eta \pm \epsilon$ is of the desired kind. In one of the two continua $f(x) - \eta$ is positive, in the other negative.

§ 2. *Existence of at most Two Continua.*

The curve C divides the plane into continua R_i, perhaps infinite in number, whose only boundary points are points of the curve. With the help of the auxiliary theorem of § 1 it may be shown that the number of continua is at most two.

AUXILIARY THEOREM II. *Each continuum R_i has every point of the curve C as a boundary point. On the other hand the (x, y)-points near any point of C belong to at most two continua, so there are in all at most two.*

It is evident that any continuum R_i must have at least one boundary point (ξ, η) on the curve C. It follows that R_i must include one of the continua into which the rectangle about (ξ, η) is divided according to the first auxiliary theorem, and must therefore have all the points of C near (ξ, η) as boundary points. Consider now the largest interval $\tau \leqq t < T$ $(T \leqq \tau + \omega)$ of parameter values all of which define boundary points of R_i. The upper boundary T can not be less than $\tau + \omega$, for the point (x, y) defined by it is a limit point of boundary points, and therefore is itself a boundary point of R_i. With the help of Auxiliary Theorem I it follows as above that all the parameter values near T also define boundary points of R_i, and the interval $\tau \leqq t < T$ with $T < \tau + \omega$ could not be the largest.

§ 3. *Existence of at least Two Continua.*

The existence of at least two continua is shown by means of a function $N(a, b)$ which is constant as long as (a, b) remains in one of the continua R_i, but which takes at least two different values at points of the plane not on the curve C. This function is the integral

$$(2) \qquad N(a, b) = \int_t^{t+\omega} \frac{(\phi - a)\psi' - (\psi - b)\phi'}{r^2}\, dt \qquad [r^2 = (\phi - a)^2 + (\psi - b)^2]$$

whose primitive is the function u defined by the equations

$$\cos u = \frac{\phi - a}{r}, \quad \sin u = \frac{\psi - b}{r}.$$

It is evident from (2) that $N(a, b)$ is a continuous function of a and b when (a, b) is not on the curve C, and on account of the periodicity of ϕ and ψ its value is always some multiple of 2π. Since any two points in the same continuum R_i can be joined by a curve along which N varies continuously, the values of N at points in the same region must be always equal to the same multiple of 2π.*

In order to show that $N(a, b)$ has at least two different values in the plane, consider again a point (ξ, η) near which the curve C has the form (1), and let $h > 0$ be so chosen that the only point of C on the ordinate $x = \xi$ between $\eta - h$ and $\eta + h$ is (ξ, η). Denote by N, N_1, u, u_1 and r, r_1 the values of $N(a, b)$, the angles, and the radii corresponding to the two points $a = \xi$, $b = \eta \pm h$. The difference $N - N_1$ is an integral whose primitive is the function $u - u_1$ defined by the equations

$$\cos (u - u_1) = \frac{(\phi - \xi)^2 + (\psi - \eta)^2 - h^2}{rr_1},$$

$$\sin (u - u_1) = - \frac{2h(\phi - \xi)}{rr_1}.$$

On account of the periodicity of ϕ and ψ the values of $u - u_1$ at t and $t + \omega$ differ by a multiple of 2π. Furthermore, as t varies through an interval ω, $u - u_1$ passes once and only once through the value π. For when $u - u_1 = \pi$,

$$\phi - \xi = 0, \quad \frac{(\psi - \eta)^2 - h^2}{rr_1} = -1, \quad \therefore\ |\psi - \eta| < h,$$

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* The quotient $N/2\pi$ has been called by Ames the "order" of the point (a, b); loc. cit., p. 353. The proof given in this section is similar to his, and to one about to be published by Osgood in his Lehrbuch der Funktionentheorie, vol. 1, p. 136.

and the point (ϕ, ψ) must be the point (ξ, η), the only point of the curve C on the ordinate $x = \xi$ between $\eta - h$ and $\eta + h$. That $u - u_1$ actually passes through the value π follows because $\phi - \xi$ and consequently $\sin(u - u_1)$ change sign when t passes through the value τ. It is evident, then, that the values of $u - u_1$ at t and $t + \omega$ differ by $\pm 2\pi$, and hence

$$N - N_1 = \pm 2\pi.$$

AUXILIARY THEOREM III. *The function $N(a, b)$ is a constant in any continuum R_i and takes at least two different values at points (a, b) not on the curve C. There exist therefore at least two continua R_i.*

The theorem which was the object of the present paper follows at once from the last two auxiliary theorems:

PRINCIPAL THEOREM. *Any Jordan curve having the properties stated in §1 divides the plane into two continua, an interior and an exterior.*

§4. *Extension to Curves with Corners and Singular Points.*

The proof given in the preceding sections can be extended without much difficulty to curves having a finite number of points where the conditions (*c*) and (*d*) of §1 are not both satisfied. In order to effect the extension it is only necessary to show that a rectangle with the properties described in Auxiliary Theorem I can be constructed at the exceptional points as well as at the others. The further steps in the proof remain the same.

The exceptional points admitted are points where conditions (c) and (d) of §1 do not both hold true, but it is supposed that the direction of the tangent defined by the equations

$$\cos a = \frac{\phi'}{\sqrt{\phi'^2 + \psi'^2}}, \quad \sin a = \frac{\psi'}{\sqrt{\phi'^2 + \psi'^2}},$$

approaches definite limiting values as the parameter t approaches, from either direction, the value τ defining the exceptional point. The two limits corresponding to the two directions of approach are not necessarily the same.

Such points include corner points where all the conditions of §1 are satisfied except that ϕ' and ψ' are discontinuous, and singular points where both ϕ' and ψ' are zero provided that higher derivatives of ϕ and ψ exist which do not vanish simultaneously at the value τ.

Suppose that the X-axis is taken so as not to be perpendicular to any of the limiting directions of the tangent at exceptional points. Then there is an interval $\tau - \delta_1 \leqq t < \tau$ in which $\cos a$ and ϕ' are different from zero, although at $t = \tau$ the derivative ϕ' may vanish. In such an interval x is a monotonic function of t, varying from ξ_1 to ξ. Conversely t, and therefore also y, is a single-valued continuous function of x in the interval $[\xi_1, \xi]$. In a similar way y may be expressed as a function of x in an interval $[\xi, \xi_2]$ corresponding to parameter values between τ and $\tau + \delta_2$. Let these two functions be denoted by $y = f_1(x)$ and $y = f_2(x)$ respectively.

There are two cases to be considered. If ξ_1 and ξ_2 include the value ξ between them, then y is a single-valued continuous function of x in the whole interval $[\xi_1, \xi_2]$ and the rectangle can be constructed exactly as in § 1. If ξ_1 and ξ_2 are both on the same side of ξ, then the intervals $[\xi_1, \xi]$ and $[\xi, \xi_2]$ overlap, but $f_1(x)$ is always different from $f_2(x)$ in the common interval. The construction of the rectangle is the same except that in this case the two arcs of C cut the rectangle on the same ordinate. One of the two continua into which the rectangle is divided consists of points (x, y) for which x is in the interval common to $[\xi_1, \xi]$ and $[\xi, \xi_2]$, and y is between the corresponding values of $f_1(x)$ and $f_2(x)$. The other points of the rectangle form the other continuum.

If a curve has a finite number of exceptional points of the kind described above, but elsewhere satisfies the condition of § 1, *it divides the plane into two continua, an interior and an exterior.*

PRINCETON UNIVERSITY,
December, 1905.

DETERMINATION OF ASSOCIATED SURFACES.

BY DR. BURKE SMITH.

(Read before the American Mathematical Society, October 28, 1905.)

It is here proposed to develope a set of formulas for the cartesian coordinates of surfaces which are associate to a given surface referred to its asymptotic lines. The coordinates of an associated surface may thus be found directly from those of the given surface, by differentiation and substitution, when one has solved an equation of the Laplace type.

Since the images on the Gauss sphere of the asymptotic lines on any surface S_0 are the same as the images of a conjugate system of lines on the associated surface S, we may find S by determining the surface which has, as images of a conjugate system, the lines on the sphere which are the images of the asymptotic lines on S_0. Suppose S_0 is referred to its asymptotic lines, and let e, f, g represent the fundamental magnitudes of the corresponding sphere. Then in order that the lines $u =$ const., $v =$ const., on the sphere should be the images of the asymptotic lines on S_0 it is necessary and sufficient that

$$(1) \qquad \frac{\partial}{\partial u}\begin{Bmatrix}1\,2\\1\end{Bmatrix}' = \frac{\partial}{\partial v}\begin{Bmatrix}1\,2\\2\end{Bmatrix}',$$

where the symbols of Christoffel are formed for the sphere. To find the coordinates of a surface S which has the above lines as images of a conjugate system, we proceed as follows: If X, Y, Z are the direction cosines of the normals to S along the conjugate lines, the cartesian coordinates x, y, z of S may be obtained by solving the equations *

$$(2) \qquad \begin{cases} xX + yY + zZ = W, \\ x\dfrac{\partial X}{\partial u} + y\dfrac{\partial Y}{\partial u} + z\dfrac{\partial Z}{\partial u} = \dfrac{\partial W}{\partial u}, \\ x\dfrac{\partial X}{\partial v} + y\dfrac{\partial Y}{\partial v} + z\dfrac{\partial Z}{\partial v} = \dfrac{\partial W}{\partial v}, \end{cases}$$

where W is a solution, linearly independent of X, Y, Z, of the equation,

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* Bianchi-Lukat, Vorlesungen über Differential-Geometrie, p. 139.

$$(3)\qquad \frac{\partial^2 W}{\partial u \partial v} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' \frac{\partial W}{\partial u} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}' \frac{\partial W}{\partial v} + fW = 0.$$

Now (3) has equal invariants by virtue of (1). If we put $W = \theta\rho^{-\frac{1}{2}}$, where $-\rho^{-2} = K_0$ is the curvature of S_0, and take account of Codazzi's equations, which in this case are

$$\frac{\partial \log \rho}{\partial u} = -2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}', \quad \frac{\partial \log \rho}{\partial v} = -2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}',$$

we may write (3) in the form

$$(4)\qquad \frac{\partial^2 \theta}{\partial u \partial v} \left(\frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial u \partial v} - f \right) \theta.$$

X, Y, Z are particular solutions of (3), and putting $X = \xi/\sqrt{\rho}$, $Y = \eta/\sqrt{\rho}$, $Z = \zeta/\sqrt{\rho}$, ξ, η, ζ will be particular solutions of (4). Solving (2) for x, y, z and remembering that $\sqrt{eg - f^2} = [X, \partial Y/\partial u, \partial Z/\partial v]$, we have,

$$(5)\qquad \begin{gathered} x = \frac{[W, \partial Y/\partial u, \partial Z/\partial v]}{\sqrt{eg - f^2}}, \quad y = \frac{[X, \partial W/\partial u, \partial Z/\partial v]}{\sqrt{eg - f^2}}, \\ z = \frac{[X, \partial Y/\partial u, \partial W/\partial v]}{\sqrt{eg - f^2}}. \end{gathered}$$

Now by the formulas of Lelieuvre, the coördinates x_0, y_0, z_0 of S_0 satisfy six relations of the form

$$(6)\qquad \frac{\partial x_0}{\partial u} = \zeta \frac{\partial \eta}{\partial u} - \eta \frac{\partial \zeta}{\partial u}, \quad \frac{\partial x_0}{\partial v} = \eta \frac{\partial \zeta}{\partial v} - \zeta \frac{\partial \eta}{\partial v},$$

etc., and since $\rho^2 = \xi^2 + \eta^2 + \zeta^2$ we may write in place of (5)

$$(7)\qquad \begin{cases} x = \dfrac{\theta \dfrac{\partial^2 x_0}{\partial u \partial v} - \dfrac{\partial \theta}{\partial u}\dfrac{\partial x_0}{\partial v} - \dfrac{\partial \theta}{\partial v}\dfrac{\partial x_0}{\partial u}}{\rho^{\frac{3}{2}}\sqrt{eg - f^2}} = \dfrac{\theta \dfrac{\partial^2 x_0}{\partial u \partial v} - \dfrac{\partial \theta}{\partial u}\dfrac{\partial x_0}{\partial v} - \dfrac{\partial \theta}{\partial v}\dfrac{\partial x_0}{\partial u}}{\xi \dfrac{\partial^2 x_0}{\partial u \partial v} - \dfrac{\partial \xi}{\partial u}\dfrac{\partial x_0}{\partial v} - \dfrac{\partial \xi}{\partial v}\dfrac{\partial x_0}{\partial u}} \\[2ex] y = \dfrac{\theta \dfrac{\partial^2 y_0}{\partial u \partial v} - \dfrac{\partial \theta}{\partial u}\dfrac{\partial y_0}{\partial v} - \dfrac{\partial \theta}{\partial v}\dfrac{\partial y_0}{\partial u}}{\rho^{\frac{3}{2}}\sqrt{eg - f^2}} = \dfrac{\theta \dfrac{\partial^2 y_0}{\partial u \partial v} - \dfrac{\partial \theta}{\partial u}\dfrac{\partial y_0}{\partial v} - \dfrac{\partial \theta}{\partial v}\dfrac{\partial y_0}{\partial u}}{\eta \dfrac{\partial^2 y_0}{\partial u \partial v} - \dfrac{\partial \eta}{\partial u}\dfrac{\partial y_0}{\partial v} - \dfrac{\partial \eta}{\partial v}\dfrac{\partial y_0}{\partial u}} \end{cases}$$

$$z = \frac{\theta\frac{\partial^2 z_0}{\partial u\partial v} - \frac{\partial\theta}{\partial u}\frac{\partial z_0}{\partial v} - \frac{\partial\theta}{\partial v}\frac{\partial z_0}{\partial u}}{\rho^2\sqrt{eg-f^2}} = \frac{\theta\frac{\partial^2 z_0}{\partial u\partial v} - \frac{\partial\theta}{\partial u}\frac{\partial z_0}{\partial v} - \frac{\partial\theta}{\partial v}\frac{\partial z_0}{\partial u}}{\zeta\frac{\partial^2 z_0}{\partial u\partial v} - \frac{\partial\zeta}{\partial u}\frac{\partial z_0}{\partial v} - \frac{\partial\zeta}{\partial v}\frac{\partial z_0}{\partial u}}$$

where θ is a solution of (4) which is linearly independent of ξ, η, ζ. These equations give the coordinates x, y, z of the associated surface S from those of S_0 when θ is known.

As an example, consider the sphere. The asymptotic lines on the sphere are its imaginary generatrices, and when referred to these lines its equations take the form

$$x_0 = a\left(\frac{1+uv}{u+v}\right), \quad iy_0 = a\left(\frac{1-uv}{u+v}\right), \quad z_0 = a\left(\frac{v-u}{u+v}\right).$$

By direct calculation we have,

$$ds^2 = \frac{-4a^2}{(u+v)^2}du\,dv, \quad K = \frac{1}{a^2}, \quad \rho = ia.$$

Equation (4) becomes in this case, $\partial^2\theta/\partial u\partial v = 2\theta/(u+v)^2$, of which the general solution is

$$\theta = 2(U+V)/(u+v) - U' - V',$$

where U and V are arbitrary functions of u and v respectively and the primes denote derivatives. Substituting in (7) we have, as the equations of the associate of a sphere,

$$x = \tfrac{1}{2}a(U - uU' + \tfrac{1}{2}(u^2-1)U'') + \tfrac{1}{2}a(V - vV' + \tfrac{1}{2}(v^2-1)V'')$$
$$y = \tfrac{1}{2}ia(U - uU' + \tfrac{1}{2}(u^2+1)U'') + \tfrac{1}{2}ia(V - vV' + \tfrac{1}{2}(v^2+1)V'').$$
$$z = \tfrac{1}{2}a(uU'' - U') - \tfrac{1}{2}a(vV'' - V').$$

These are the equations of the minimal surfaces.

Consider also the pseudosphere whose equations in terms of the parameters of the asymptotic lines are

$$(8) \qquad \begin{cases} x_0 = i\operatorname{csch}(u+v)\cos(u-v), \\ y_0 = i\operatorname{csch}(u+v)\sin(u-v), \\ z_0 = u + v - \coth(u+v). \end{cases}$$

For the square of the lineal element we have

$$ds^2 = du^2 + 2[\operatorname{csch}^2(u+v) + \coth^2(u+v)]du\,dv + dv^2$$

and $K = -\rho^{-2} = -1$.

Equation (4) becomes in this case,

$$\partial^2\theta/\partial u\partial v = [\operatorname{csch}^2(u+v) + \coth^2(u+v)]\,\theta.$$

To every solution θ of this equation, which is linearly independent of ξ, η, ζ, we have from (7)

$$(9)\quad \begin{cases} x = \theta\coth(u+v)\cos(u-v) + \frac{1}{2}(\theta_u + \theta_v)\cos(u-v) \\ \qquad\qquad - \frac{1}{2}(\theta_u - \theta_v)\tanh(u-v)\sin(u-v), \\ y = \theta\coth(u+v)\sin(u-v) + \frac{1}{2}(\theta_u + \theta_v)\sin(u-v) \\ \qquad\qquad + \frac{1}{2}(\theta_u - \theta_v)\tanh(u-v)\cos(u-v), \\ z = i\theta + \frac{1}{2}i(\theta_u + \theta_v)\cos(u+v), \end{cases}$$

where the subscripts denote partial derivatives. From (8) we have $\left\{\begin{matrix}1\;2\\1\end{matrix}\right\} = \left\{\begin{matrix}1\;2\\2\end{matrix}\right\} = 0$. But since $\left\{\begin{matrix}1\;2\\1\end{matrix}\right\} = -\left\{\begin{matrix}1\;2\\1\end{matrix}\right\}'$ and $\left\{\begin{matrix}1\;2\\2\end{matrix}\right\} = -\left\{\begin{matrix}1\;2\\2\end{matrix}\right\}'$ we have for (9), $\left\{\begin{matrix}1\;1\\2\end{matrix}\right\} = 0$ and $\left\{\begin{matrix}2\;2\\1\end{matrix}\right\} = 0$. Hence (9) are the equations of surfaces of Voss, since the parametric lines consist of two families of geodesics which form a conjugate system.*

If the general solution θ of any equation of the form $\partial^2\theta/\partial u\partial v = M\theta$ is known, where M is any function of u and v, and one can find three particular solutions ξ, η, ζ linearly independent of θ, the equations of a surface S_0 referred to its asymptotic lines may be found by quadrature from (6). The surfaces which are associated to S_0 may be found by substitution in (7). Thus if one can solve an equation of the above type and can choose the three particular solutions ξ, η, ζ linearly independent of θ, the cartesian coordinates of pairs of associated surfaces may be found by six quadratures and substitution in (7). As an example, the writer has already considered the equation $\partial^2\theta/\partial u\partial v = 0$ elsewhere.†

Consider also the equation

$$\partial^2\theta/\partial u\partial v = 2\theta/(u+v)^2$$

of which the general solution is

$$\theta = 2(U+V)/(u+v) - U' - V'.$$

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* See Voss, *Sitzungsberichte der Bayrischen Akademie*, 1888, pp. 95–102.

† BULLETIN, Jan. 1906.

As three particular solutions, take

$$\xi = 2U/(u+v) - U', \quad \eta = 2V_1/(u+v) - V_1',$$
$$\zeta = 2V_2/(u+v) - V_2'.$$

Substitution in (6) gives for S_0 in this case,

$$x_0 = 2(V_2 V_1' - V_1 V_2')/(u+v) + \int (V_1' V_2'' - V_2' V_1'')dv,$$
$$y_0 = 2(V_2 U' + V_2' U)/(u+v) - U' V_2',$$
$$z_0 = -2(V_1 U' + V_1' U)/(u+v) + U' V_1'.$$

Substitution in (7) gives at once the equations of the group of associated surfaces.

The method above outlined for determining associated surfaces may be applied to the problem of determining surfaces which admit of continuous deformation with preservation of conjugate lines, since such surfaces are the associates of those whose curvature in terms of the parameters of the asymptotic lines is of the form $K = [\phi(u) + \chi(v)]^{-2}$.* In that case (4) may be written in the form

$$\frac{\partial^2\theta}{\partial u \partial v} = \left(-\frac{1}{4\rho^2}\frac{\partial\rho}{\partial u}\frac{\partial\rho}{\partial v} - f\right)\theta.$$

PURDUE UNIVERSITY,
December, 1905.

NOTE ON THE PRACTICAL APPLICATION OF STURM'S THEOREM.

BY J. E. WRIGHT, M.A.

IT is strange that the following small point is not mentioned in the text-books. The only remark I can find bearing on it is in Encyklopädie der Mathematik I., 1, 4, page 417. Suppose $f, f_1, \cdots, f_n$ are the successive Sturmian functions in *e. g.*, the case when f has only simple roots. If all the real roots of f_r are known, then we can separate the roots by using $f, f_1, \cdots, f_r$ only. For let a, b be two consecutive roots of f_r. Then between a and b f_r does not vanish, and therefore no alteration in the number of changes of sign can arise from the functions $f_r, \cdots, f_n$.

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* Bianchi-Lukat, l. c., p. 337.

Hence the number of roots of f between a and b is determined from $f, \cdots, f_r$ only. Practically to avoid zeros of f_r, we substitute $a + \epsilon$, $b - \epsilon$ in the first $r + 1$ functions, when ϵ is small. To complete the work it is necessary to discover whether or not any of the roots of f_r are roots of f, and to apply the theorem to all the intervals given by $\pm\infty$ and the roots of f_r.

Example :

$$f(x) = x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1,$$

$$f_1(x) = 5x^4 - 20x^3 + 27x^2 - 18x + 5, \quad f_2(x) = x^3 - x.$$

Roots of f_2 are -1, 0, $+1$; of these $+1$ is a root of $f(x)$.

	$-\infty$	$-1-\varepsilon$	$-1+\varepsilon$	$-\varepsilon$	$+\varepsilon$	$+1-\varepsilon$	$+1+\varepsilon$	$+\infty$	
f	−	−	−	−	−	+	−	+	Altogether three real roots.
f_1	+	+	+	+	+	−	−	+	
f_2	−	−	+	+	−	−	+	+	
	no roots		no roots		one root		one root		

Bryn Mawr,
January, 1906.

THE MOVEMENT FOR REFORM IN THE TEACHING OF MATHEMATICS IN PRUSSIA.

Ueber eine zeitgemässe Umgestaltung des mathematischen Unterrichts an den höheren Schulen. F. Klein. Leipzig, 1904, pp. 82.

Beiträge zur Frage des Unterrichts in Physik und Astronomie an den höheren Schulen. Vorträge gehalten von Behrendsen, Bose, Riecke, Stark, und Schwarzschild. E. Riecke. Leipzig, 1904, pp. 108.

Neue Beiträge zur Frage des mathematischen und physikalischen Unterrichts an den höheren Schulen. F. Klein und E. Riecke. Teil I. Leipzig, 1904, pp. 190.

Verhandlungen der Breslauer Naturforscher-Versammlung über den naturwissenschaftlichen und mathematischen Unterricht an den höheren Schulen. Leipzig, 1905, pp. 77.

Bericht der Unterrichts-Kommission der Gesellschaft deutscher Naturforscher und Aerzte über ihre bisherige Tätigkeit. Leipzig, 1905, pp. 57.

These works are recent manifestations of an important movement for the improvement of the teaching of mathematics in Prussia, which has been in active progress for quite a number of years.

At the head of the movement stands Professor Klein, of Göttingen, whose views as to the reform needed are expressed in the first work of the above list. This contains five lectures delivered by Klein at a vacation course for teachers held at the University of Göttingen in the spring of 1904. In addition to these there are reprinted several previous publications of Klein's; and a paper by E. Götting on the aim of mathematical instruction in the higher schools (15 pages). The second work contains lectures on topics relative to physics and astronomy delivered at the same course for teachers, and the third consists of the first two bound in one volume.

At the outset, Klein defends himself against two accusations that are usually made against university professors who take active interest in the problems of the schools: first, that they wish to increase the requirements in mathematics; second, that they have in mind exclusively preparation for university work in mathematics. Klein denies emphatically that he takes either of these attitudes. He wishes no raising of the mathematical level in the schools, but rather its horizontal displacement. He is not concerned about what mathematics the prospective student of mathematics in the university may learn in the schools; this student will be taken care of when he gets to the university. It is the non-mathematician that needs special attention — the prospective physician, chemist, jurist. The primary duty of the schools is to give these a mathematical equipment that shall be really fruitful in their subsequent careers.

Klein advocates, not an enlargement of the content of elementary mathematics, but a horizontal displacement of its boundary. Purists define elementary mathematics as the totality of those parts of mathematics in which the concept of limit is not used. This admits much of the theory of numbers, for example, into elementary mathematics, but excludes $\sqrt{2}$ and π. A second definition admits the idea of limit but not in the special form denoted by the symbols dy/dx and $\int y dx$. This would exclude from elementary mathematics the very common ideas of tangent to an arbitrary curve, of its length and area, of velocity, acceleration, etc., and would admit much that is not within the range of the school, for example, an "elementary theory of analytic functions of a complex variable," which, with Weierstrass, avoids Cauchy's theorem and bases all proofs on comparison of infinite power series. A third definition calls everything elementary in geometry which relates to the geometry of Euclid and the ancients. This would admit into elementary geometry the difficult modern researches on the foundations of geometry, and would exclude the simplest modern considerations on variability and transformation of figures; that is, it would keep the spirit of modern geometry entirely out of the schools.

The only definition of elementary mathematics that will be of any use to the school must be a practical one:

In all domains of mathematics those parts are to be called elementary which can be understood by a pupil of average ability without long continued special study.

The content of elementary mathematics, under this definition, is naturally dependent on the time. Improved presentation brings new topics into the elementary field. There can be no question that the determination of the derivative and the integral of x^n (n a positive integer) to-day belongs in the elementary field, and Klein's chief thesis is:

The introductory development of the function concept, the first introduction to analytic geometry, the beginning of the differential and integral calculus, should be a part of the course of every pupil.

The explanation and discussion of this thesis constitutes the backbone of the series of addresses. The early and continued use of graphic representation of simple functions is urged; for example of

$$y = ax + b, \quad y = x^2, \quad y = 1/x$$

in "Untersekunda" (age 14–15), together with varied applications, especially to the mathematical discussion of physical phenomena. As a minimum, the law of falling bodies, the motion of the pendulum and wave motion should be understood mathematically. The treatment must be sufficiently formal to assure real understanding, but the main end is clear grasp of the fundamental concepts and their concrete significance.

"It is often said that the beginning of higher mathematics (*i. e.*, of the differential and integral calculus) means a revolution in the thinking of the student. The plan of instruction which I am urging is meant to spare the student this revolution by accustoming him from the beginning to the processes of thought which are later to prove fruitful. This 'revolution' is merely a test of the unsuitability of the earlier work to the later aims."

It would be interesting to cite many other instructive details of these addresses. In fact the whole work is well worth translating into English, but what has been said above must suffice for the present writing, with the remark in conclusion that here, as elsewhere, Klein also takes occasion to lay stress on the demand that the universities give special attention to the needs of prospective teachers of mathematics. It will be recalled that three years' university study of mathematics are a part of the professional preparation of every teacher of secondary mathematics in Prussia. Consequently, it is likely that prospective teachers constitute the majority of the mathematical students at the universities, and yet until recently the latter have given no explicit attention to their needs. The subject matter and the methodology of elementary mathematics deserve special attention from the university, and Klein announces for the winter next following a course by himself for prospective teachers under the title: "Concerning mathematical instruction in the higher schools."

The fourth and fifth reports relate to activities of the German society of natural scientists and physicians, dating from the year 1901, when the so-called "Hamburg theses" relative to instruction in the biological sciences were presented. In the consideration of these theses at the session at Cassel in 1903, it was decided, upon motion of Professor Klein, to make the entire field of instruction in the natural sciences and mathematics the subject of thorough discussion. This discussion took place at Breslau in 1904 and No. 4 of the above list is the published report of this discussion. Among the formal papers presented were one by Fricke on "The present status of instruction in the natural sciences and mathematics in the higher schools" and one by Klein under the title "Remarks on instruction in mathematics and physics." Here also Klein urged the importance of the function concept and of the elements of the calculus in the mathematical side of modern culture, the need of considering mathematics in closer relation with its applications in the exact sciences and in all phases of life capable of precise formulation, and the danger of allowing the formal logical side to predominate too strongly in instruction in mathematics. Both speakers lay marked stress on the importance of thorough preparation of teachers, and emphasize the duty of the universities in this connection, not only in the matter of scientific equipment, but also in that of methodic and didactic preparation. Klein also advocates that teachers be given regularly a semester's leave of absence to give them opportunity to survey the progress of their science, and by personal relations and inspection on travels here and there, to make acquaintance with the progress that is being made in this field, and especially to see what part their own subject is playing in the culture of our day.

At the close of this discussion, a commission of twelve was appointed, including Klein and other mathematicians, to draw up courses of study in mathematics and the natural sciences, bringing into unified and practical form the various propositions made in the discussion. The report of this commission is the fifth work named above, and an educational document of great significance.

After a general outline by the chairman, Professor A. Gutzmer, of the University of Jena, of the work of the commission, its leading recommendations and the principles which governed it in making them, there follow detailed curricula in each of the subjects mathematics, physics, chemistry, botany, and zoology. Only that in mathematics concerns us here. Not undervaluing logical training, the commission regarded strengthening of space intuition and training to the habit of functional thinking as the most important task of instruction in mathematics. As the goal to be reached by this instruction there were set up a scientific survey of the organization of the whole field covered in the school, a certain facility of regarding things mathematically, and above all insight into the importance of mathematics in the exact knowledge of nature, and in the culture of the day in general.

In accordance with this general point of view, the detailed curriculum gives new prominence to the concrete phases of mathematics (drawing, graphic methods), to the function concept and the representation of functions by curves, and to the applications of mathematics. The introduction of the elements of the calculus was left to the option of the teacher.

Though relating directly only to German conditions, these publications are of significance beyond the borders of that country. The first and the last will prove of most interest to the general reader. The movement itself, of which all these works are the outgrowths, is undoubtedly of the first importance. In addition to active interest on the part of leading mathematicians and of the strong and influential society appointing and supporting the commission whose report we have considered, other evidences of a widespread and significant participation in the movement are not lacking. Of these only the representation of the Deutsche Mathematiker-Vereinigung on the Commission, and the space devoted to pedagogic matters in its *Jahresbericht*, need be mentioned here. While this movement is directly concerned only with mathematics in Germany, it is nevertheless of international significance, both on account of the fundamental nature of the changes which it proposes and the weight attaching to the names of the societies and men that are engaged in it. In its essentials, it is in close harmony with movements of the day for improvement of the teaching of mathematics in England, in France and in the United States, which agree in demanding:

1. That the presentation take more careful account of the pupils' powers; that it be psychological as well as logical; that consequently the intuitional, the inductive, the concrete, be given prominent place in the teaching of mathematics.
2. That the amount of complex or merely technical work be diminished.
3. That, in addition to its disciplinary value, the importance of mathematics for a clear understanding of nature, and the fundamental character of mathematics in the fabric of modern civilization be brought effectively before the pupils.

J. W. A. YOUNG.

THE UNIVERSITY OF CHICAGO,
January 24, 1906.

VECTOR ANALYSIS.

Vorlesungen über die Vektorenrechnung. Von E. JAHNKE. Leipzig, B. G. Teubner, 1905. xii + 235 pp.

THE entering class at an American college is more or less prepared on elementary geometry and algebra, and perhaps on trigonometry. The problem of collegiate instruction has, therefore, one fairly well-defined premise. All the students who continue their work in mathematics, for no matter what purpose, must study analytic geometry and calculus. From this point on, however, the needs of different students are different, and to a considerable extent the aims and the subjects of instruction must vary. What shall be the place, and what the meaning of further instruction in algebra?

It is possible to proceed in several ways. In the first place the plan usually followed may be adopted and more algebra, similar to that with which the student is already familiar, may be taught. On the other hand the questions of the foundations of algebra and arithmetic may be studied with the object of showing the student the meaning of postulates, the significance of the number system, and the fact that algebra is merely symbolic and devoid of any inherent concrete interpretation. The present fervor for logic may make this course of procedure popular and in the future much time may be devoted to this branch of algebra. The result, however, might turn out to be far from justifying the expenditure in time and exertion. It is difficult to get any real grasp of the meaning of postulates or any considerable enjoyment in the logical development of a subject from them, unless the consequences of other related systems of postulates be also investigated. Thus it would seem better to postpone this second extension of algebra until such time as similar questions in geometry with especial reference to Lobatchevskian and Riemannian geometry shall be taken up. In the third place it may be the aim of the instructor to broaden the student's ideas on the general subject of algebra, to bring out the natural generalization of those ideas, to show how geometry and algebra may be united into a geometric algebra somewhat in contrast to algebraic or analytic geometry.

It seems as if this third way of continuing algebra, no more difficult of execution and considerably more practical than the preceding two, is all too much neglected here in America and in the world at large. The method of advance almost universally followed in respect to algebra is as if on entering college the student were put to work studying the nature of postulates and the formal side of geometry or developing further the synthetic methods of higher geometry instead of proceeding to analytic geometry and through it to the generalizations of modern geometry. If a thorough course (or even two) in analytics — an amount considerably in excess of that required for elementary calculus and its applications — be presented before matters of postulates and higher synthetic methods, why should not some corresponding training in geometric algebra come before the investigation of those other algebraic questions? The reason is perhaps more historical than rational. And although it would doubtless be gross exaggeration to state that geometric algebra is on a par with analytic geometry as regards its general importance, yet in view of its great usefulness in geometry and its present widespread use in physics, not to mention its own intrinsic interest, it certainly deserves a greater relative recognition than is now given to it.

To further this end nothing seems more propitious than the appearance of Jahnke's Lectures on vector analysis. The title is but poorly descriptive of the book, which is far wider in its scope and method. These lectures are really lectures on multiple algebra and form an excellent introduction to the subject — no more than an introduction because the theory of the linear function which is of such fundamental importance is hardly mentioned and the closely allied theory of linear associative algebra is not touched at all. The author's method of treatment is logical and pedagogic from beginning to end. He first considers in detail the algebra of the plane and only later proceeds to corresponding discussions for space. The similarities between the two cases render the study of space much easier after the plane has been treated; and the dissimilarities bring out very clearly the changes that must still be made in order to generalize to space of n dimensions. What is especially remarkable for a German text-book is that not merely does the author work out numerous examples but he actually leaves a larger number as exercises to be worked out by the student. This feature of the book will certainly appeal to any one who may wish to teach this subject, in which collections of problems are hard to find.

In the first chapter the beginnings of Grassmann's point analysis, the addition and subtraction of points affected with numerical coefficients, is taken up. This leads at once to the free vector as the difference of two points and to the addition and subtraction of vectors. The third chapter discusses localized vectors and the multiplication of points. Next comes the multiplication of free vectors followed by a large and various set of applications to problems in mechanics and physics. A sixth chapter on regressive multiplication completes the treatment of the plane. From this brief indication of the contents of the first eighty-eight pages of the Lectures, it can be seen that the reader has been brought step by step to the different and manifold ideas and methods underlying geometric algebras of the non-linear-associative type. The study of the complex variable will furnish a correspondingly simple case of this latter type.

Proceeding to space, the author follows exactly the same program, weaving together side by side the point analysis and the vector analysis. Here, however, there are two classes of directed quantities (vectors and bivectors or lines and planes) and the product of three vectors or four points is a scalar; whereas in the plane there was only a single kind of vector and the product of two vectors was a scalar. Among the applications may be mentioned the theory of moments of inertia, of the screw with Chasles's theorems, and of the null-system. Further there is a treatment of differentiation, about thirty pages in the whole, less than ten of which are on divergence, curl, and Gauss's theorem. Stokes's theorem seems not to be mentioned. This insignificant amount of attention paid to space differentiation and integration, which makes the book much less useful than many another to the physicist, leaves greater room for the purely algebraic and geometric side and is, therefore, particularly to be commended from the standpoint of one wishing to learn or to teach multiple algebra as a continuation of elementary algebra and of a little analytic geometry.

There is one point on which a little adverse criticism or at any rate a word of caution seems necessary. On page 110 the author says: "In most books on vector analysis it is a question of vector and scalar products, not of outer and inner products. While the inner and scalar products are identical, there is an essential difference between the outer and vector products. If $[\mathbf{ab}]$ be an outer product, $[\mathbf{ab}]$ is a vector of higher dimensions than $\mathbf{a}$ or $\mathbf{b}$ individually. On the other hand $V\mathbf{ab}$ or $\mathbf{a} \times \mathbf{b}$ is a simple vector $\mathbf{c}$. In this kind of multiplication the conception of dimensions is lost." In a paper read at the congress at Heidelberg, I have analysed this argument in detail and come to the conclusion that if it be not wrong it is, at least from the point of view of the physicist, so inadequate as to be worthless.* There is no need of repeating the analysis here. The trouble arises through the use of the term dimensions in two different senses. There is just one fundamental proposition underlying the idea of physical dimensions, namely, that the physical dimensions of a product are the product of the dimensions of the individual factors. This is independent of the analysis by which such product is represented. It is even independent of the kind of product. Thus if **a** and **b** are vectors representing length, the scalar product **a** · **b**, the inner product [**a**|**b**], the vector **a** × **b**, the bivector [**ab**], the dyad **ab**, the quaternion **ab**, all have the dimensions $[L]^2$. It will not do for the physicist to lose sight of the fact for a moment. The followers of Grassmann, however, use the term dimension in a more geometric sense. If **a**, **b**, **c** be vectors representing lengths, **ab** is a bivector or vector of the second class (zweiter Stufe) and **abc** is a trivector, a scalar, a quantity of the third class. Again, by the law of regressive multiplication, (**abc**)**a** becomes a vector of the first class. As a matter of fact the physical dimensions of (**abc**)**a** are $[L]^4$. As a geometer one may take great delight in the theory of geometric dimensions in multiple algebra, but as a physicist he must discard it as confusing. This is precisely what is done by the physicists who use Grassmann's notation.

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* "On products in additive fields." Verhandlungen des III. internationalen Mathematiker-Kongresses in Heidelberg, 1904, § 5, pp. 209–211.

The whole matter may, however, be let pass with good grace inasmuch as the book is otherwise singularly free from blemishes and well fulfils the requirements for an elementary text on geometric or multiple algebras. Jahnke's lectures supplemented first with the elementary geometric theory of imaginary numbers and of quaternions and second with Gibbs's theory of dyadics, where the ideas of products are considerably more extended than anywhere else, would form an excellent introductory course on higher algebra and would furnish, in addition, a very respectable knowledge of some parts of projective geometry, mechanics, and physics.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
January, 1906.

CELESTIAL MECHANICS.

An Introduction to Celestial Mechanics. By F. R. MOULTON, Ph.D., Instructor in Astronomy in the University of Chicago. New York, The Macmillan Company, 1902. xv + 384 pp.

THE need of a work written in the English language which would properly introduce astronomical students into the subject of celestial mechanics has been keenly felt for some time. In many of our universities the teaching of astronomy is confined to practical and to what is commonly called theoretical astronomy, the latter involving chiefly the derivation of suitable formulas for the numerical solution of parabolic, elliptic, and hyperbolic orbits and of special perturbations. As a result of this, the young astronomer often enters upon his work as an observer without a proper understanding of the general problems of astronomy to the solution of which his observations are to contribute, or as a mechanical computer without a real insight into the methods which he is applying, or both. It is probably not far from true that the sum total of the knowledge in celestial mechanics of many American students at the close of their university career is confined to the subject matter of Watson's Theoretical astronomy, supplemented sometimes only by a mechanical working familiarity with the excellent collections of formulas in von Oppolzer's Lehrbuch zur Bahnbestimmung. The importance of this subject is considered so great that two works along these lines have appeared in Germany in rather close succession: a new edition by Dr. Buchholz, Göttingen, of Klinkerfuss's Theoretische Astronomie, and Bauschinger's Bahnbestimmung, just off the press. All of these serve an excellent, though very special purpose, and are priceless in the hands of the teacher, but if properly absorbed, they perhaps take more than their just proportion of the student's time. They are, unfortunately, not directly designed to develop a taste nor the necessary preparation for the study of the methods of investigation and the results attained in celestial mechanics, such as would inspire to the study of Tisserand's Mécanique céleste, or Poincaré's Les nouvelle méthodes de la mécanique céleste, or Brown's Lunar theory, or other standard works along the same lines.

In the opinion of the writer, Dr. Moulton has succeeded in filling this gap and has exhibited rare skill in the selection and treatment of the necessary material. But, to quote from the preface, the work will also be welcomed by mathematical students who "desire to get an idea of the processes by means of which astronomers interpret and predict celestial phenomena." Moulton's work will develop in the astronomical student a desire for the study of the advanced mathematics necessary for further pursuit of celestial mechanics, and in the mathematical student a greater love for his subject through the practical applications to which he sees it put. It will serve, no doubt, to draw the astronomer and mathematician into closer relation. The book contains for its size an abundance of material, including the results of some of the author's own investigations; the treatment is original, clear and concise. Mention should be made in this connection of the method adopted in Chapter VII in developing Lagrange's particular solutions of the problem of three bodies; of his investigations regarding the validity of the expressions for the elements α_i and β_i in power series in m_1 and m_2 (pages 266 and 257); of his general method of determining the elements of an orbit of a comet or planet after the solution of the heliocentric and geocentric distances is accomplished (article 236), etc.

A feature of the work to be particularly commended on account of the breadth which it gives to the subjects treated is the inclusion of applications to astrophysical problems, and of principles useful in geodesy. The importance of the study of celestial mechanics is thus brought home to the student specializing in astrophysics or geodesy. There are articles on the velocity from infinity applied to the escape of atmospheres (Chapter II, Article 36); the heat of the sun, the temperature of meteors, the meteoric theory of the sun's heat, Helmholtz's contraction theory (Chapter II, articles 39–43); Gegenschein (Chapter VII, Article 124).

At the end of each of the eleven chapters is given a carefully selected set of problems, the solution of which will add materially to a clear understanding of the text, and an historical sketch and bibliography which goes to prove a thorough familiarity with the literature of the subject, although in places omissions may be noted of references of at least equal importance to those given. The work is strictly up to date, touching as it does on the most recent researches of Hill, Poincaré, Darwin and others.

In the last two chapters of the book the author attempts to derive the working formulas necessary for the computation of a parabolic or general orbit of a comet or planet. This attempt is a radical departure from the otherwise excellent plan of the book. Although a mere outline of these methods might have been in place, the last two chapters could have been entirely omitted without in the least impairing the excellency of the work. Why should not Hansen's method of general perturbations be given a chapter, if it is considered worth while to consider in detail such very special topics as the adaptation of the solution of the problem of two bodies to the numerical computation of an orbit? In the first place, in the brief space which the author has devoted to the subject, it is not possible to do this subject justice so as to be of actual practical use to the student. The inclusion also of formulas for the reduction from apparent to mean place, etc., without proof, encourages mechanical computing. Such matters really have no place in an introduction to celestial mechanics. The presentation is patterned somewhat after Watson's Theoretical astronomy, excellent certainly in its time but now improved upon by other works. Take, for instance, the discussion on page 344 of the various fundamental formulas: why not follow Oppolzer's plan of merely stating that six conditions being available for the determination of five elements, one observation is made incomplete by substituting for it the condition that the middle place shall lie in an arbitrary auxiliary circle passed through it; that the formulas become simplest when the great circle is passed through the solar place at the middle date, and that in the so-called "Ausnahmefall" the most accurate result is obtained by adopting an auxiliary great circle through the middle place perpendicular to the geocentric motion. This latter is Oppolzer's method, and does not fail for parabolic orbits. Similarly, Oppolzer's method for determining elliptic orbits is certainly to be preferred to the original method of Gauss owing to its higher degree of approximation. But I consider these matters of really too special a nature to have a place in Moulton's work. As said before, they do not give the astronomical student a real insight into the nature of the numerical problem, nor can they be of great interest to the mathematical student.

Chapter I is designed to leave no doubt in the student's mind regarding the foundations of the subject. There might be a doubt as to whether or not the reader ought to be familiar with most of the elementary principles considered, but a review of them on the basis of Moulton's logical presentation can do no harm.

Chapter II deals with rectilinear motion and applications referred to above. It includes the consideration of forces varying directly as the distance, inversely as the square of the distance, proportional to the velocity, and proportional to the square of the velocity, the general method of solution being given the place it deserves.

Chapter III is on central forces. After a consideration of simultaneous differential equations, the orbits corresponding to a given law of force are discussed, and vice versa. The proper foundation is then laid for a consideration of the universality of

Newton's law on the basis of observed double star orbits, which leads to the conclusion that if the attraction does not depend upon the direction of the bodies from each other, Newton's law of gravitation operates in binary systems.

Chapter IV contains a judicious selection of topics relating to the potential attraction of bodies, and should particularly appeal to the geodesist. Chapter V is on the problem of two bodies. Chapter VI gives the fundamental principles of the problem of n bodies.

Chapter VII is on the problem of three bodies. Its subject matter is particularly well chosen, leaving the reader with a clear conception of all that has been accomplished in attempting its solution. The particular solutions are clearly presented, the treatment of the surfaces of zero velocities, the stability of particular solutions, etc., is concise and to the point.

Chapter VIII contains the geometric interpretation of perturbations and is valuable as a means of assisting the student to a proper understanding of the effects of perturbation on the elements of an orbit.

Chapter IX treats of analytic methods of determining perturbations. The object of the chapter is stated by the author in the following words: "The chief difficulties which arise in getting an understanding of the theories of perturbation come from the large number of variables which it is necessary to use, and the very long transformations which must be made in order to put the equations in a form suitable for actual computations. It is not possible, because of the lack of space, to develop in detail the explicit expressions adapted to computation; and indeed it is not desired to emphasize this part, for it is much more important to get an accurate understanding of the nature of the problem, the mathematical features of the methods employed, the limitations which are necessary, the exact places where approximations are introduced, if at all, and their character, the origin of the various sorts of terms, and the foundations upon which the celebrated theorems regarding the stability of the solar system rest" (page 256). The author has performed his task in an exceptionally satisfactory manner.

Chapters X and XI contain the derivation of formulas for the computation of the orbits of comets and asteroids, and have been referred to above. A. O. LEUSCHNER.

UNIVERSITY OF CALIFORNIA,
BERKELEY ASTRONOMICAL DEPARTMENT,
February 10, 1906.

SHORTER NOTICES.

Géométrie, Premier et Second Cycle. Par EMILE BOREL. Paris, Librairie Armand Colin, 1905. 383 pp.

THIS is one of a series of elementary text-books written by Borel to meet the demand caused by the recent activity for the improvement of instruction in elementary mathematics in France. As the author states, it has been his aim to make a book more in line with the ideas of the geometers and analysts of the nineteenth century who have taught that "Geometry is the study of groups of movements." With this end in view he has introduced the ideas of symmetry and displacements as often as possible, thereby making many demonstrations simpler and more concrete.

The book is divided into three parts. Part one contains about what is ordinarily called books I and II. The treatment however is quite distinctive. The theorems concerning isosceles triangles and parallel lines are proved by the use of symmetry. The symmetry of regular figures is discussed at some length. For use in constructions, the square is introduced as a third instrument and many constructions are thereby made much simpler. Part two is the extension of the first part to space. The ideas of translation and rotation are used quite freely. Part three treats similitude, areas and volumes. The trigonometric functions of the angle are given and used in the demonstration of many of the theorems which follow. The book closes with a short discussion of conics and a chapter devoted to the approximate determination of plane areas. A good collection of problems is given at the end of each part.

C. L. E. MOORE.

Orthogonale Axonometrie. Ein Lehrbuch zum Selbststudium. Mit 29 Figurtafeln in besonderem Hefte. By Dr. RUDOLF SCHÜSSLER, Associate Professor of Mathematics in the Technical School at Graz. Leipzig, Teubner, 1905. 170 pp.

IN this book the attempt has been made to develop axonometry as an independent method of representation without the use of the horizontal and vertical planes of descriptive geometry and without the assumption of any previous knowledge of conical or parallel perspective. The method of procedure is rather similar to that of Pelz* but made much more concrete and ele-

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*See the Vienna *Sitzungsberichte*, vols. 81, 83, 90 and the Prague *Sitzungsberichte*, 1885, 1895.

mentary, as well as more comprehensive and systematic. The book begins very concretely; after defining what axonometry is, the projection of the rectangular coordinates of a point, the projections of a line on each of the coordinate planes, and the intersections of a plane with each of the coördinate planes are explained with excellent clearness. The discussions regarding a finite but inaccessible point of intersection of two lines, and of a line and plane (pages 19, 20) are particularly commendable. Each particular or exceptional case is discussed in sufficient detail so that every case met in subsequent constructions is a direct application of the principles already explained.

The treatment of pyramids and prisms is now easy and natural; a thorough treatment of shades and shadows is added, applying to all bodies bounded by planes. The section applied to metrical properties, lengths, normals, angles, etc., is less direct and natural than the preceding ones and the question may well be asked whether for such purposes this method of representation has any advantage over the ordinary h, v method.

The circle is treated with care and the usual detail; a large number of alternate proofs are given for the same theorem, perhaps too many for a student who is reading it alone — and for such students the book was designed — though they are valuable to one already familiar with the projection for purposes of comparison. Now follow conics, cones, cylinders and their curves of intersection, and the ideas here developed are applied to surfaces of revolution. The only slip noticed is on page 153, where it is stated that an inflexion in the meridian curve will produce a corner in the shadow cone. The shadow cone must touch the parallel circle generated by the point of inflexion.

Numerous exercises for the student to solve are added to each section, many of them provided with a figure for suggestion. Notice is always given on the margin when a new figure is considered. Finally, mention should be made of the figures themselves, which were drawn by the author. They are very well done; proper lines are drawn heavier or lighter to show at a glance just what the figure means. The whole work shows care and thoughtfulness on every page, and is provided with an excellent index. It is an important addition to the literature on constructive drawing.

VIRGIL SNYDER.

NOTES.

The Thirteenth Summer Meeting and Fifth Colloquium of the American Mathematical Society will be held at Yale University during the entire week, September 3–8, 1906. The first two days will be devoted to the regular sessions for the presentation of papers. The Colloquium, which will open on Wednesday morning, will include the following courses of lectures: Professor E. H. Moore, "On the theory of bilinear functional operators"; Professor Max Mason, "Selected topics in the theory of boundary value problems of differential equations"; Professor E. J. Wilczynski, "Projective differential geometry."

The nineteenth regular meeting of the Chicago Section of the American Mathematical Society will be held at the Northwestern University, Evanston, Ill., on Saturday, April 14, 1906, the first session opening at 10 A. M.

At the meeting of the London mathematical society held on February 8, the following papers were read: By Major P. A. MacMahon, "Partitions of numbers in space of two dimensions"; by Dr. H. F. Baker, "The Eisenstein-Sylvester extension of Fermat's theorem."

The National academy of sciences will hold its annual meeting at Washington, D. C., April 16–18.

The American association for the advancement of science will hold a summer session at Ithaca, New York, during the week June 26 to July 2. Dr. Edward Kasner is vice-president and Professor L. G. Weld secretary of Section A.

The foreign universities named below offer advanced courses in mathematics during the summer semester of 1906 as follows:

University of Halle.—By Professor G. Cantor: Theory of numbers, four hours; Seminar, two hours. — By Professor A. Wangerin: Differential geometry, five hours; Definite integrals and differential equations, four hours; Selected chapters of potential theory, one hour; Seminar, two hours. — By Professor A. Gutzmer: Differential calculus with exercises, five hours; Theory of functions, four hours; Selected chapters of analytic mechanics, one hour; Seminar, two hours. — By Professor V. Eberhard: Algebra, I, four hours; Analytic geometry of conics, two hours. — By Professor F. BERNSTEIN: Historical review of the principal fields of pure mathematics, two hours; Mathematics of insurance, two hours.

UNIVERSITY OF JENA. — By Professor R. HAUSSNER: Differential and integral calculus with exercises, five hours; Plane analytic geometry, four hours; Division and quadrature of the circle, two hours; Seminar, one hour. — By Professor J. THOMAE: Elliptic functions, four hours; Mathematical geography, four hours. — By Professor G. FREGE: Analytic mechanics II., four hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Applications of the calculus, four hours; Seminar, one hour. — By Profsssor A. MAYER: Analytic dynamics, five hours; with exercises, one hour. — By Professor O. HÖLDER: Applications of elliptic functions, three hours; Selected chapters of elliptic modular functions, two hours; Seminar, one hour. — By Professor K. ROHN: Curves of order three and four, four hours; Determinants, two hours; Seminar, one hour. — By Professor F. HAUSDORFF: Ordinary differential equations, four hours; with exercises, one hour. — By Professor H. LIEBMANN: Plane analytic geometry, four hours; Introduction to algebraic analysis, four hours; with exercises, one hour.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Integral calculus, five hours; Substitutions and higher algebraic equations, four hours; Mechanics of non-rigid bodies, two hours; Seminar in mechanics, one and one half hours. — By Professor A. VOSS: Introduction to the theory of differential equations, four hours; Analytic geometry of space, five hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Definite integrals, four hours; Applications of elliptic functions, two hours. — By Professor K. DOEHLEMANN: Descriptive geometry (axonometry, perspective), three hours; with exercises, two hours; Synthetic geometry, II, four hours; Imaginary elements in geometry, one hour. — By Professor E. v. WEBER: Determinants, four hours; Differential calculus, four hours; with exercises, two hours. — By Professor H. BRUNN: Elements of higher mathematics, four hours. — By Dr. F. HARTOGS: Selected chapters of the theory of functions, four hours.

UNIVERSITY OF ROSTOCK. — By Professor O. STAUDE: Analytic geometry of space, four hours; Analytic mechanics, four hours; Seminar, two hours.

UNIVERSITY OF WÜRZBURG. — By Professor F. PRYM: Integral calculus, six hours; Proseminar, two hours; Seminar, two hours. — By Professor E. SELLING: Analytic mechanics, four hours; Spherical astronomy, two hours. — By Professor G. ROST: Analytic and synthetic geometry of conics, four hours; Applications of the calculus to plane curves, four hours; Twisted curves and surfaces, four hours; Noneuclidean geometry, two hours; Seminar in pure mathematics, two hours; Seminar in mechanics, two hours.

UNIVERSITY OF BERN. — By Professor J. H. GRAF: Spherical harmonics, three hours; Bessel functions, three hours; Definite integrals, three hours; Differential and integral calculus, two hours; Theory of investments, two hours; Seminar, two hours. — By Professor E. OTT: Differential calculus, two hours; Analytic mechanics, I, two hours. — By Professor G. HUBER: Elliptic integrals with applications, two hours. — By Dr. C. MOSER: Insurance, two hours. — By Dr. J. V. PEXIDER: Theory of numbers, three hours; Elements of the theory of aggregates, one hour; Analytic theory of numbers, two hours; Prime numbers, one hour. — By Dr. L. CRELIER: Synthetic geometry of space, two hours; Geometry of the triangle, with exercises, four hours.

UNIVERSITY OF ZÜRICH. — By Professor H. BURKHARDT: Algebraic analysis, four hours; Mathematical theory of dissipative phenomena, four hours; Seminar, two hours. — By Professor A. WEILER: Analytic geometry, II, four hours; Descriptive geometry, II, four hours; Synthetic geometry, two hours. — By Dr. E. GABLER: Principal theorems of differential and integral calculus, two hours; Political arithmetic, two hours; Content and method of instruction in geometry in preparatory schools, one hour.

DURING the fortnight beginning April 19 a vacation course for gymnasium teachers will be held at the University of Göttingen. Professors Klein and Behrendsen will explain in detail the schedules for mathematics and physics proposed by the commission of the society of German scientists and physicians (see the review of these reports by Professor J. W. A.

Young in the BULLETIN, pages 347–352). Regular courses of lectures will be given by Professors Behrendsen, Prandtl, Runge, Simon, Voigt and Wagner.

BEGINNING on Friday, March 23, Professor H. A. LORENTZ, of the University of Leiden, will give at Columbia University a course of eight lectures on "The theory of electrons and its applications to the phenomena of light and radiant heat." These lectures will be given on Fridays from 4 to 6 p. m. and on Saturdays from 10 to 12 a. m., March 23 to April 7, and from 4 to 6 p. m. on Wednesday, April 11, and Thursday, April 12.

THREE senior wranglers in mathematics sit in the British court of appeal — Fletcher Moulton, just elevated to the bench, Lord Justice Ronier and Lord Justice Sterling. Mathematicians are also strongly represented at present among the high court judges; the Lord Chief Justice Buckley and Justices Channell, Joyce and Bray were all wranglers, and Justice Sutton was a senior optime.

THE Smith's prizes for the present year have been awarded to C. F. RUSSELL, of Pembroke College, for his essay "The geometric interpretation of apolar binary forms," and to F. J. M. STRATTON, of Gonville and Caius college, for his essay "A problem in tidal evolution suggested by the motion of Saturn's ninth satellite."

THE exposition of arts and industries held in Brussels in the fall of 1905 awarded its gold medal to the periodical *l'Enseignement mathématique* for its services to mathematical pedagogy.

PLANS are being perfected for preparing a suitable memorial and monument to the late Professor GUIDO HAUCK in the rooms of the technical school at Berlin, at which he was professor for twenty-seven years and thrice rector. A general invitation for contributions has been issued, signed by representatives of most of the academies, universities and technical schools of Germany. Remittances may be sent to Herr Kiesel, Sekretär der Technischen Hochschule, Berlin-Charlottenburg.

THE mathematical society of Berlin announces that the hindrances which prevented the erection of a suitable monument over the grave of WEIERSTRASS no longer exist and that the stone will be placed in position this spring. Friends and students of Weierstrass who wish to contribute for this purpose may send their contributions to Professor Dr. J. Knoblauch, Berlin W, Karlsbad 12.

DR. E. ZERMELO, of the University of Göttingen, has been promoted to an assistant professorship of mathematics.

DR. F. HARTOGS has been appointed docent in mathematics at the University of Munich.

PROFESSOR G. HETTNER, of the technical school at Berlin, has been awarded the order of the crown of the third class by the German emperor.

DR. L. BOULANGER has been appointed professor of mechanics at the University of Lille.

DR. M. F. W. DYSON, of the observatory at Greenwich, has been appointed professor of astronomy at the University of Edinburgh.

DR. R. FUCHS has been appointed docent in mathematics, and Dr. REISSNER has been promoted to an associate professorship of mechanics at the technical school of Berlin.

MR. E. T. WHITTAKER, of Cambridge, has been appointed Andrews professor of astronomy at the University of Dublin, and astronomer royal of Ireland.

PROFESSORS T. J. I'A. BROMWICH, of Queen's College, Galway, and J. H. JEANS, of Princeton University, have been elected fellows of the Royal society of London.

PROFESSOR W. SCHEIBNER, of the University of Leipzig, celebrated his eightieth birthday, January 8.

COLONEL I. H. WING, of Batfield, Wisconsin, an alumnus of Bowdoin College, has given the college $50,000 to endow the chair of mathematics.

THE Royal astronomical society has awarded its gold medal to Professor W. N. CAMPBELL, of the Lick Observatory, for his researches on the motion of stars in the direction of the line of sight.

DR. E. B. WILSON has been promoted to an assistant professorship of mathematics at Yale University.

PROFESSOR W. J. HUSSEY, of Lick Observatory, has been appointed professor of astronomy at the University of Michigan.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMES (L. D.). An arithmetic treatment of some problems in analysis situs. (Diss., Harvard.) Baltimore, Friedenwald, 1905. 4to. (*American Journal of Mathematics*, Vol. 27, pp. 343–380.)

ANGUS (A. H.). A preliminary course in differential and integral calculus. London, Longmans, 1906. 8vo. 116 pp. Cloth. 2s. 6d.

DÖLP (H.). Grundzüge und Aufgaben der Differential- und Integralrechnung nebst den Resultaten. Neubearbeitet von E. Netto. 11te Auflage. Giessen, Töpelmann, 1905. 8vo. 4 + 216 pp. Cloth. M. 1.80

DUCCI (E.). Sezioni del cono retto circolare e deduzione delle loro principali proprietà; lezioni. Melfi, Grieco, 1905. 8vo. 49 pp.

FIELD (P.). On the form of a plane quintic curve with five cusps. 1906. 4to. (*Transactions of the American Mathematical Society*, Vol. 6, pp. 26–32.)

GRAF (J. H.). Beiträge zur Biographie Jakob Steiners. Mit einem unbekannten Portrait Steiners. Bern, 1905.

HARDY (G. H.). The integration of functions of a single variable. Cambridge, University Press, 1905. 8vo. 8 + 53 pp. (Cambridge tracts in mathematics and mathematical physics, No. 2.)

JAHN (O.). Einiges vom Zahlbegriff. Halle, 1905. 4to. 16 pp. M. 1.00.

KEYSER (C. J.). Mathematical emancipations; the passing of the point and the number three; dimensionality and hyperspace. 1906. 8vo. (*The Monist*, Vol. 16, pp. 65–83.)

LUBELZA (D. J.). Lecciones de cálculo infinitesimal. Vol. I: Diferencial (escalar y vectorial.) Madrid, Ratés, 1905. 8vo. 16 + 128 pp. Fr. 7.50.

MACCOLL (H.). Symbolic logic and its applications. London, Longmans. 1906. 8vo. Cloth. 4s. 6d.

NETTO (E.). See DÖLP (H.).

PIRONDINI (G.). Nuovo metodo per costruire delle funzioni continue a spazi lacunari; memoria. Parma, Rossi-Ubaldi, 1905. 8vo. 16 pp.

SOUCHON (A.). Sur l'application des fonctions elliptiques au calcul des perturbations planétaires. Tours, Deslis, 1905. 8vo. 20 pp.

VITALI (G.). Sugli ordini di infinito delle funzioni reali; nota. Bologna, Gamberini, 1905. 8vo. 9 pp.

WILTZ (H.). Mathematische Unterrichtsbriefe. Unter Mitwirkung von bewährten Fachgelehrten bearbeitet und herausgegeben. Erster Lehrbrief und erster Wiederholungsbrief. Strassburg, Wolstein, 1906. 30 + 10 pp. 8vo. M. 1.50

II. ELEMENTARY MATHEMATICS.

BOREL (E.). Algèbre (second cycle). 3e édition, revue, corrigée et augmentée. Paris, Colin, 1905. 16mo. 7 + 403 pp. Fr. 3.00

——. Géométrie, à l'usage des écoles normales primaires (programmes de 1905), des écoles primaires supérieures, des aspirants et aspirantes aux brevets de l'enseignement primaire, et des lycées et collèges de jeunes filles. Paris, Colin, 1905. 16mo. 10 + 383 pp. Fr. 2.75

——. Géométrie (cours de mathématiques, rédigé conformément aux nouveaux programmes) (premier et second cycles). Paris, Colin, 1905. 16mo. 10 + 383 pp. Fr. 3.00

CHAMBER. Mathematical tables, consisting of logarithms of numbers from 1 to 108,000, trigonometrical, nautical and other tables. New edition, edited by J. Pryde. New York, Van Nostrand, 1906. 8vo. 454 pp. Cloth. $1.75

CRACKNELL (A. G.). See WORKMAN (W. P.).

D'ARCY (R. F.). New trigonometry for beginners. London, Methuen, 1905. 8vo. 92 pp. Cloth. 2s. 6d.

FOWLER (C. W.). Inductive geometry; an introduction to and preparation for the regular text. Louisville, Fowler, 1905. 16mo. 8 + 52 pp. Cloth. $0.50

FRANCHIS (M. DE). Elementi di aritmetica razionale ad uso delle scuole secondarie superiori. Palermo, Sandron, 1905. 16mo. 188 pp. L. 2.00

GRÉVY (A.). Géométrie, à l'usage des élèves des classes de cinquième B et troisième B. 4e édition. Paris, Vuibert, 1906. 16mo. 323 pp. Fr. 2.25

——. Géométrie plane, à l'usage des élèves des classes de seconde C et D. Paris, Vuibert, 1906. 8vo. 296 pp. F. 3.00

GUTZMER (A.). See REFORMVORSCHLÄGE.

HOČEVAR (F.). Lehr- und Uebungsbuch der Geometrie für Untergymnasien. 7te Auflage. Unveränderter Abdruck der 6ten Auflage. Wien, Tempsky, 1905. 2 + 122 pp. M. 1.70

LAZZERI (G.). Manuale di trigonometria piana. 2a edizione. Livorno, Giusti, 1906. 16mo. 6 + 124 pp. L. 1.00

MÜLLER (H.) und PLATH (J.). Lehrbuch der Mathematik zur Vorbereitung auf die Mittelschullehrerprüfung und auf das Abiturientenexamen am Realgymnasium. Leipzig, Teubner, 1905. 8vo. 8 + 236 pp. M. 4.00

PAPELIER (G.). Supplément au précis d'algèbre et de trigonométrie, à l'usage des élèves de mathématiques spéciales, conforme au programme du 26 juillet 1904. Paris, Vuibert, 1906. 8vo. 110 pp.

PETIT formulaire mathématique et technique (extraits des cours de l'Ecole spéciale des travaux publics, à Paris). 2e édition. Paris, 1906. 16mo. 59 pp.

PIERPONT (A. E.). Elements of geometry in theory and practice. Based on report of committee appointed by the Mathematical Association, 1902. Part 1, comprising subject-matter of Euclid, Book I, 1–34. London, Longmans, 1906. 8vo. 164 pp. Cloth. 2s.

PLATH (J.). See MÜLLER (H.).

PRYDE (J.). See CHAMBER.

RAFFALLI (A.). Découverte de la quadrature du cercle. Bastia, Piaggi, 1905. 8vo. 8 pp.

Reformvorschläge für den mathematischen und naturwissenschaftlichen Unterricht, entworfen von der Unterrichtskommission der Gesellschaft deutscher Naturforscher und Aerzte. Nebst einem allgemeinen Bericht über die bisherige Tätigkeit der Kommission, von A. Gutzmer. Leipzig, Teubner, 1906. 8vo. 6 + 48 pp.

Schultz (A.). Advanced algebra. New York, Macmillan, 1906. 12mo. 14 + 562 pp. Half leather. $1.25.

Utescher (O.). Der Rechenunterricht an höheren Lehranstalten. Freiburg, 1905. 8vo. 45 pp.

Workman (W. P.) and Cracknell (A. G.). Geometry, theoretical and practical. Part 1. London, Clive, 1906. 8vo. 368 pp. Cloth. 3s. 6d.

Wortner (F.). Geometrie und geometrisches Zeichnen für Knabenbürgerschulen. 2te, im wesentlichen unveränderte, in der neuen Rechtschreibung durchgeführte Auflage. Wien, Tempsky, 1905. 8vo. 176 pp. M. 1.80

III. APPLIED MATHEMATICS.

Änding (E.). See Encyklopädie.

Bohny (F.). Theorie und Konstruktion versteifter Hängebrücken. Leipzig, Engelmann, 1905. 8vo. 6 + 109 pp. M. 5.00

Caspari (C. E.). See Encyklopädie.

Chollett (J.). Traité de géométrie descriptive. Première partie, à l'usage des élèves des classes de première C et D. 3e édition. Paris, Vuibert, 1906. 8vo. 188 pp. Fr. 2.00

——. Traité de géométrie descriptive et Compléments de géométrie. Deuxième partie, à l'usage des élèves des classes de mathématiques A et B et des candidats à l'Ecole de Saint-Cyr. 2e édition. Paris, Vuibert, 1906. 8vo. 204 pp. Fr. 2.50

Church (I. P.). Hydraulic motors, with related subjects, including centrifugal pumps, pipes, and open channels; designed as a text-book for engineering schools. New York, Wiley, 1905. 8vo. 6 + 269 pp. Cloth. $2.00

Cohn (F.). See Encyklopädie.

Colardeau (E.). Approximations dans les mesures physiques et dans les calculs numériques qui s'y rattachent. Paris, 1905. 8vo. Fr. 5.00

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THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE ninth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at Stanford University, on Saturday, February 24, 1906. The following sixteen members were present:

Professor R. E. Allardice, Mr. A. J. Champreux, Professor R. L. Green, Professor M. W. Haskell, Mr. Edwin Haviland, Professor L. M. Hoskins, Professor D. N. Lehmer, Dr. J. H. McDonald, Dr. W. A. Manning, Professor G. A. Miller, Professor H. C. Moreno, Dr. T. M. Putnam, Mr. Arthur Ranum, Professor Irving Stringham, Professor A. W. Whitney, Professor E. J. Wilczynski.

The attendance also included a number of teachers of mathematics and physics who are not members of the society. A morning and an afternoon session were held, Professor Allardice acting as chairman at both sessions. It was agreed to hold the next meeting at the University of California on September 29, 1906.

The following papers were read at this meeting:

(1) Dr. J. H. MCDONALD: "The theory of the reduction of hyperelliptic integrals of the first kind and of genus 2 to elliptic integrals by a transformation of the nth order."

(2) Dr. W. A. MANNING: "On multiply transitive groups."

(3) Mr. ARTHUR RANUM: "A new kind of congruence group and its application to the group of isomorphisms of any abelian group" (preliminary report).

(4) Professor D. N. LEHMER: "On the orderly listing of substitutions."

(5) Professor D. N. LEHMER: "Note on the values of z of given modulus which give maximum or minimum values to the modulus of a given rational integral function of z."

(6) Professor R. E. ALLARDICE: "Note on Legendre's equation."

(7) Professor R. E. ALLARDICE: "On the multiple points of unicursal curves."

(8) Professor E. J. WILCZYNSKI: "Outline of a projective differential geometry of curved surfaces" (preliminary report).

(9) Mr. E. T. BELL: "Method of dealing with the problems connected with prime numbers" (preliminary report).

(10) Dr. T. M. PUTNAM: "Theorems on perfect numbers."

(11) Dr. J. H. MCDONALD: "A method of simultaneous approximation to two consecutive roots of an algebraic equation of degree n all whose roots are real."

(12) Dr. J. H. MCDONALD: "Remarks on the calculation of roots of Bessel functions."

(13) Professor M. W. HASKELL: "On collineations" (preliminary report).

(14) Professor G. A. MILLER: "Groups in which every subgroup of composite order is invariant."

(15) Professor G. A. MILLER: "The groups which contain exactly thirteen operators of order 2."

Mr. Bell was introduced by the Secretary of the Section. Professor Miller's second paper has appeared in the March number of the BULLETIN. Abstracts of the other papers are given below. The abstracts are numbered to correspond to the titles in the list above.

1. According to the Weierstrass-Picard theorem, when there corresponds to the algebraic relation $y^2 = R_6(x)$ (where $R_6(x)$ denotes a polynomial of the 6th degree) a single reducible integral, then a set of normal integrals may be found, both being reducible, for which the table of canonical periods is known. Dr. McDonald examined the problem in purely algebraic formulation and established a theorem which concerns the relations between the two reducible integrals and shows how to construct either when the other and its reducing substitution are known.

2. If a doubly transitive group of degree n has $r+1$ transitive subgroups $H, H_1, \cdots, H_r$, of degrees $q, q+q_1, \cdots, q+q_1+\cdots+q_r$, less than n, it is possible, as Dr. Manning proves, to group the letters of H q_i by q_i in systems of imprimitivity in at least

$$q_i\left(1+\frac{1}{q_r}+\frac{1}{q_{r-1}}+\cdots+\frac{1}{q_i}\right)$$

distinct ways, for $i = 1, 2, \cdots, r$. This is a generalization on the theorems of Jordan and Marggraff on primitive groups with transitive subgroups of lower degree. A number of more special theorems on multiply transitive groups are given in this paper.

3. Mr. Ranum proved that the group of isomorphisms I of any abelian group G of order p^m having n invariants can be expressed as a linear congruence group in n variables, such that each row (or column) of a matrix has its own modulus, namely the corresponding invariant. A general formula is obtained for the order of any such group I. Its series of composition is found; and if $p > 3$, I is shown to be solvable when, and only when, no two of its invariants are equal, while if $p = 2$ or 3, I is solvable when no three of its invariants are equal.

Corresponding to every characteristic subgroup H of G, it is shown that I has an invariant subgroup J whose isomorphisms transform all the operators of H into themselves and whose matrices are congruent to identity when the moduli are the invariants of H. Moreover, corresponding to the characteristic quotient group H' of the same type as H, I has an invariant subgroup J' of the same type as J, whose matrices are obtained from those of J by interchanging rows and columns.

Finally a still more general class of linear congruence groups is mentioned, in which each coefficient of a matrix has its own modulus, and it is shown that if all the moduli in the same rows are equal, the group is necessarily the group of isomorphisms of some abelian group.

4. Professor Lehmer discussed a notation by means of which a substitution is determined by a single number and the remarkable laws arising from the notation in the resulting multiplication tables of various groups were brought out. A method was explained for constructing the multiplication table of the symmetric group on any number of letters with no computation.

5. In Professor Lehmer's second paper certain theorems were brought out concerning the family of curves into which a system of concentric circles are transformed by means of a rational integral function $f(z)$. In particular the locus of the feet of normals from the origin on this family of curves was discussed and the theorem proved that not only the curve passes through the root points of $f(z) = 0$ but that the line joining a root point of $f(z) = 0$ to the origin is tangent to this locus at that point or else forms part of the locus. The paper will be offered to the *Annals of Mathematics* for publication.

6. The equation referred to by Professor Allardice is the linear differential equation

$$(1-x^2)y''-2xy'+n(n+1)y=0.$$

In Forsyth's Differential Equations (third edition, pages 155 and following), a method is given for finding a relation between the two solutions of this equation, by putting $y=uv-w$, where v stands for one of the solutions. The expression obtained for u is $\frac{1}{2}\log\{(x+1)/(x-1)\}$, and w is obtained in infinite series. This solution is stated to be valid in all cases, except the critical cases where $2n$ is a positive or negative odd integer; though there seems to be no good reason for excluding these cases, if the solution is valid in all other cases. Another form of w is obtained as a sum of functions P_r; but this solution is restricted to the case where n is a positive integer. Now the supposed solution vanishes identically in every case, except when n is a positive integer. In the latter case, certain terms become indeterminate; if these be excluded, a correct solution is obtained, but if they be included, the expression vanishes identically. In the expression in terms of functions P_r, only one term becomes indeterminate, and the limit of this term is Q_n; and thus the expression for Q_n in terms of functions P_r is obtained.

7. The second paper by Professor Allardice was concerned with the calculation of the multiple points of a curve when the coördinates of the curve are given explicitly as rational functions of a parameter. A simple proof was first given of the known theorem that the double point of a cubic is given by the hessian (or canonizant) of a certain binary cubic. By the same method an equation of the sixth degree was then obtained for the three double points of a quartic, in the form of a determinant of the third order. A quartic curve in space was next considered, and the condition for the existence of a double point was shown to be the vanishing of the invariant I_3 of a certain binary quartic. From this condition another form of the equation for the multiple points of a plane quartic was deduced, namely the equation obtained by equating to zero a certain canonizant of two binary quartics. The canonical forms of the quartics obtained by means of this canonizant are $(A, B, C, D, E)(x, y)^4$ and $(A', \lambda B, \lambda C, \lambda D, E')(x, y)^4$.

8. The system of differential equations

$$(1)\quad y_{uu} + 2ay_u + 2by_v + cy = 0,\quad y_{vv} + 2a'y_u + 2b'y_v + c'y = 0,$$

where $y_u = \partial y/\partial u$, $y_v = \partial y/\partial v$, $y_{uu} = \partial^2 y/\partial u^2$, etc., has four linearly independent solutions, which may be taken as homogeneous coördinates of a point on a surface. The curves $u =$ const. and $v =$ const. will be asymptotic curves upon this surface. Professor Wilczynski has begun the investigation of the invariants and covariants of (1), which theory, on its geometric side, is equivalent to a projective differential geometry of surfaces.

If the independent variables are not transformed, while y is put equal to $\lambda\bar{y}$, a new system is obtained from (1), satisfied by $\bar{y}$. The invariant formations (semi-invariants and semi-covariants) for this transformation are

$$(2)\quad a',\ b,\ e = a_v - b'_u,\ f = c - a_u - a^2 + 2bb',\ g = c' - b'_v - b'^2 + 2aa'$$

and

$$(3)\quad \begin{aligned} &y,\ z = y_u + ay,\quad \rho = y_v + b'y,\\ &\sigma = y_{uv} + b'y_u + ay_v + \tfrac{1}{2}(a_v + b'_u + 2ab')y. \end{aligned}$$

The invariants are functions of the quantities (2) left unchanged by arbitrary transformation of the independent variables. a', b, and e are themselves such invariants. All of them may be expressed in terms of these three and of

$$(4)\quad h = b^2(f - b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2,\ k = a'^2(g - a'_u) - \tfrac{1}{4}a'a'_{vv} + \tfrac{5}{16}a'^2_v$$

by the help of certain operations involving differentiation processes.

The geometric interpretations are based upon the author's theory of ruled surfaces. In fact, there is associated with every asymptotic curve the ruled surface made up of the asymptotic tangents of the second set along this curve. For example, if $e = 0$, the osculating hyperboloids of the two ruled surfaces, which thus correspond to a given point of the given surface, coincide. If a' or b vanishes, the surface itself is ruled.

9. Mr. Bell's paper showed that the sifting of the primes from one to infinity can be made to depend on certain arith-

metical progressions, whose first terms are the squares of the odd numbers taken in order, and whose common differences are the odd numbers, also taken in order. This method gives all the primes uniformly, and can be used as a starting point from which to determine the number of primes less than a given integer. It is unnecessary to assume the knowledge of any prime number except 2 in dealing with the two problems of determining the primes from one to infinity, and the number of primes between given limits.

10. Dr. Putnam proved in his paper on perfect numbers that a number containing k distinct prime factors can not be perfect if all its primes are equal to or greater than $\frac{3}{2} k + 1$, the number being indeed deficient in this case. Odd perfect numbers being necessarily of the form $p^{\pi} A^2$, it was shown that if the primes in A all occur to the power 2, or all occur to the power four the number cannot be perfect.

11. The method of approximation developed in Dr. McDonald's second paper consists of a chain of quadratic equations depending on three arbitrary numbers, whose roots simultaneously converge to two consecutive roots of an equation of degree n all of whose roots are real.

12. Dr. McDonald's third paper was devoted to the following points:

(1) Genus of the function $J_n(x)$, Rayleigh's determination of least roots. (2) Calculation of early roots. (3) Determination of limit to value of possibly occurring common roots. (4) Relations among the polynomials deduced from recurring relations between Bessel functions.

13. Professor Haskell gave explicit formulas for reducing the collineations leaving a conicoid unchanged to Cremona substitutions in the plane, in such a way as to afford a basis for enumeration of the groups of such collineations, showing that the orders of such groups would be mn, or $2mn$, where m and n may take all values of the orders of groups of linear fractional substitutions of m variables. He developed the following theorem:

Any group leaving a conicoid unchanged is the product or twice the product of two groups of linear fractional substitutions on two independent variables; and conversely, any two such groups of linear fractional substitutions will generate a group leaving a conicoid unchanged.

14. Professor Miller's paper is devoted to a complete determination of the groups in which every subgroup of composite order is invariant but some subgroups of prime order are non-invariant. If the order of such a group is not a power of a single prime number it is of the form pq^2, where $p > 2$, $q + 1$ is divisible by p, and p, q are primes. The subgroup of order q^2 is of type (1,1) and is the only subgroup of composite order. Moreover, when a non-abelian group contains only one subgroup of composite order the order of the group is pq^2. The necessary and sufficient condition that every subgroup of composite order in a non-abelian and non-hamiltonian group of order p^m, $m > 5$, is invariant is that the group contains invariant operators of order p^{m-2}. If p is odd this condition is necessary and sufficient for every value of $m > 2$, and there are just two such groups for every value of m. When $p = 2$, there is one additional group when $m = 5$ and there is only one possible group when $m = 3$, viz., the octic group. Each of these groups of order p^m contains only one invariant subgroup of order p and has a commutator subgroup of order p. With the single exception of the given group of order 32, the group of cogredient isomorphisms is of order p^2. For the group of order 32 it is of order 16. The paper has been offered to the *Archiv der Mathematik und Physik* for publication.

G. A. MILLER,
Secretary of the Section.

AN APPLICATION OF THE THEORY OF DIFFERENTIAL INVARIANTS TO TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.

BY J. E. WRIGHT, M. A.

(Read before the American Mathematical Society, December 29, 1905.)

IT has been proved by Darboux * that a family of surfaces which makes part of a triply orthogonal system must satisfy a differential equation of the third order. This differential equation is given by Darboux, in his Leçons sur les systèmes orthogonaux, page 20, in the form $S = 0$, where S is a certain six-rowed determinant. Now, since $S = 0$ has a geometric interpretation, it is natural to expect that S is a differential invariant. If so, it must be expressible as an algebraic invariant of certain forms which may be readily written down.* In this note S is expressed as an algebraic invariant of the forms in question and certain immediate consequences are given.

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* "Sur les surfaces orthogonales" (*Bulletin de la Société philomath.*, 1866, p. 16), (*Annales de l'Ecole normale*, 1^{re} séries, vol. 3 (1866), p 97); see Leçons sur les systèmes orthogonaux, pp. 13–14 for complete bibliography.

The determinant S contains only third and lower derivatives of u with respect to x, y, and z, where $u =$ constant is the family of surfaces considered and x, y, z are rectangular cartesian coördinates. Hence the only possible algebraic forms of which it can be an invariant are

$$\sum_{r=1}^{3} \sum_{s=1}^{3} a_{rs} U_r U_s, \quad \text{and} \quad S_1,\ S_2,\ S_3,$$

where the notation of the author's paper already quoted is used. Now with the particular set of variables used, namely rectangular cartesians, the first of these forms becomes $U_1^2 + U_2^2 + U_3^2$, and the others

$$\left(U_1 \frac{\partial}{\partial x} + U_2 \frac{\partial}{\partial y} + U_3 \frac{\partial}{\partial z} \right)^{\lambda} u,$$

when λ takes the values 1, 2 and 3.

It is noticeable that u_{ik} and certain quantities A_{ik} turn up symmetrically in S. Since we have a form $\sum_i \sum_k u_{ik} U_i U_k$, it is suggested that a covariant $\sum_i \sum_k A_{ik} U_i U_k$ is required. Now †

$$A_{ik} = \sum_{l=1}^{3} (u_l u_{ikl} - 2 u_{il} u_{kl});$$

we use the ordinary symbolic notation for algebraic invariants and put

$$l_x = S_1, \quad a_x'^2 = b_x'^2 = S_2, \quad a_x^3 = S_3,$$

$$a_x^2 = b_x^2 = \sum_{r=1}^{3} \sum_{s=1}^{3} a_{rs}\, U_r\, U_s.$$

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* See a paper by the author, "The differential invariants of space," *Amer. Jour. of Math.*, vol. 27 (1905), pp. 335–336.

† Darboux, Leçons sur les systèmes orthogonaux, p. 19.

Also let

$$h_x^2 = \sum_{i=1}^{3} \sum_{k=1}^{3} A_{ik}\, U_i\, U_k,$$

then it is not difficult to show that

$$h_x^2 = (aab)\,(lab)a_x^2 - 2a'_x b'_x(a'ab)\,(b'ab),$$

provided

$$a_x^2 = U_1^2 + U_2^2 + U_3^2.$$

Hence, since h_x^2 is expressed as a covariant, we have its form for a general a_x^2. The determinant S may now be readily shown to be an invariant of the three quadratics a_x^2, h_x^2 $a_x'^2$ and the one linear form l_x. Its symbolic expression is

$$(hal)\,(ha'l)\,(aa'l).$$

This invariant has a simple geometric interpretation in connection with the ternary forms. Equated to zero it is the condition that the straight line l_x meets the three conics a_x^2, $a_x'^2$, and h_x^2 in six points in involution.

Now that S is expressed as an invariant of the algebraic forms, its expression may be obtained in terms of generalized coordinates ρ_1, ρ_2, ρ_3. The differential invariant theory shows that it is the same algebraic invariant of certain other forms which may be readily obtained. In fact, if $a_x^2 = aU_1^2 + bU_2^2 + cU_3^2 + 2fU_2U_3 + 2gU_3U_1 + 2hU_1U_2$,

$$S_1 = \sum_{i=1}^{3} U_i \frac{\partial u}{\partial \rho_i},$$

$$\Delta S_{m+1} = \begin{vmatrix} \sum_{i=1}^{3} U_i \frac{\partial S_m}{\partial \rho_i}, & F_1, & F_2, & F_3 \\ \frac{\partial S_m}{\partial U_1}, & a, & h, & g \\ \frac{\partial S_m}{\partial U_2}, & h, & b, & f \\ \frac{\partial S_m}{\partial U_3}, & g, & f, & c \end{vmatrix}$$

for $m = 1, 2$.

The expression F_i is equal to

$$\sum_{r=1}^{3}\sum_{s=1}^{3}\begin{bmatrix} r\ s \\ i \end{bmatrix} U_r U_s,$$

where

$$\begin{bmatrix} r\ s \\ i \end{bmatrix} = \tfrac{1}{2}\left\{\frac{\partial a_{ri}}{\partial \rho_s} + \frac{\partial a_{si}}{\partial \rho_r} - \frac{\partial a_{rs}}{\partial \rho_i}\right\}$$

is Christoffel's three index symbol, and Δ is the discriminant of a_x^2.

We may use the generalized expression thus obtained to find the condition that the parametric surfaces $\rho_3 =$ constant may form part of a triply orthogonal system. This condition is however too long to be given here.

As another example we may find the condition that the surfaces $u\ (x, y, z) =$ constant are all minimal. This condition, in cartesian coordinates, is

$$(u_{11} + u_{22} + u_{33})(u_1^2 + u_2^2 + u_3^2) = u_{11}u_1^2 + u_{22}u_2^2 + u_{33}u_3^2 + 2u_{23}u_2u_3 + 2u_{31}u_3u_1 + 2u_{12}u_1u_2,$$

where suffixes denote differentiations.

Translated into symbolic notation it becomes $(aa'l)^2 = 0$. Hence the condition expresses that a certain straight line cuts two conics harmonically.

If we take our fundamental quadratic form a_x^2 to be $\sum_{i=1}^{3} H_i^2 U_i^2$ and our minimal system $\rho_3 =$ constant, this invariant condition gives $H_1\ \partial H_2/\partial \rho_3 + H_2\ \partial H_1/\partial \rho_3 = 0$, and hence, of a triply orthogonal system, the parametric surfaces $\rho_3 =$ constant are minimal if $H_1 H_2$ is a function of ρ_1 and ρ_2 only, where the element of length is given by

$$ds^2 = H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2 + H_3^2 d\rho_3^2.$$

BRYN MAWR,
January, 1906.

SURFACES GENERATED BY CONICS CUTTING A TWISTED QUARTIC CURVE AND AN AXIS IN THE PLANE OF THE CONIC.

BY PROFESSOR VIRGIL SNYDER.

(Read before the American Mathematical Society, February 24, 1906.)

THE surface generated by the family of ∞^1 conics which cut five director curves is the simplest generalization of a scroll. Various formulas have been derived, particularly by Stuyvaert,* and the case of a unicursal quintic as directrix was first given by Bertini,† and further developed by Nugteren.‡ It is the purpose of this note to mention an interesting configuration to which the preceding methods do not apply.

1. Given a rational non-singular twisted quartic curve c_4, and a straight line l not cutting it. Let the planes π through l and the points P on l be in (1, 1) correspondence. A plane π will cut c_4 in four points Q_i which with P uniquely determine a conic c_2. It is required to determine the order of the surface F generated by c_2 when π describes the axial pencil about l. Let R_2 be the quadric upon which c_4 lies; let κ, κ' be the two points in which l cuts R_2, and s_3, s_3' the trisecants of c_4 passing through them respectively. The plane (l, s_3) will cut c_4 in a fourth point Q_4 and c_2 will consist of s_3 and $Q_4 P$, if it be assumed that P is not at κ. The second generator of R_2 in this plane is the line $Q_4\kappa'$ and does not therefore lie on F unless P is at κ'. Similarly, the lines s_3' and $Q_4'P'$ will make up a second conic.

Any conic c_2 cuts R_2 in four points, all lying on c_4, hence it can cut no trisecant apart from points on the quartic curve. When P is at κ, the corresponding c_2 has five points on R_2, hence lies entirely on R_2, and also when P is at κ' the conic in its plane lies on R_2. Since every trisecant of c_4 lies on R_2 it follows that every such trisecant will cut five conics of the system, hence *F is of order five.*

Since any plane through l contains only c_2 besides l, it follows that *l is a triple line on F_5.* The complete intersection of F_5

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* "Etude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre;" dissertation, Gand, 1902.

† "Sulle curve gobbe razionali del quinto ordine," in the Collectanea Mathematica in memoriam D. Chelini, Mediolani, 1881, pp. 313–326.

‡ "Rationale Ruimtekrommen van de fijde Orde;" dissertation, Utrecht, 1902.

and R_2 consists of c_4, $2c_2$, s_3 and s_3', making a configuration of order 10 having fourteen actual double points and two triple points, the latter being κ, κ'.

Other straight lines lie on F_5 also. If in any plane π two points of c_4 be collinear with P, the line joining them and the line joining the other points of c_4 in that plane will constitute the conic. To determine the number of such planes, consider the three planes ω formed by l and a bisecant of c_4 passing through P. To every position of π correspond three planes ω. Conversely, in any plane ω are four points of c_4, making six bisecants cutting l. Each bisecant determines a point P, and thus uniquely fixes π. Between ω, π exists a (6, 3) correspondence, having therefore nine coincidences. In each plane lie two lines belonging to F_5 and cutting c_4 twice, hence: *F_5 contains eighteen bisecants of c_4.*

The bisecants of c_4 which cut l define a scroll R_9 of order 9, on which l and c_4 are each triple, and s_3, s_3' are triple generators. The intersection of F_5 and R_9 is made up of l counted nine times, c_4 counted three times, the two trisecants which cut l each counted three times, and the eighteen bisecants mentioned above. This may be expressed thus:

$$(F_5, R_9) = l(9) + c_4(12) + 2s_3(6) + 18s_2(18).$$

2. In case κ and the plane (l, s_3) are corresponding elements, then the residual point in which the (degraded) c_2 cuts l is indeterminate, hence this plane is a factor of F_5. The line l is double on the other factor F_4. We now have

$$(F_4, R_2) = c_4(4) + 2s_3(2) + c_2(2),$$

$$(F_4, R_9) = c_4(12) + l(6) + 2s_3(6) + 12s_2(12).$$

There are now but six coincidences in the correspondence between ω and π, apart from s_3 which counts for three.

3. If κ, (l, s_3) and κ', (l, s_3') are both pairs of corresponding elements, the surface is a cubic, on which l is a simple line. The equations are

$$(F_3, R_2) = c_4(4) + 2s_3(2),$$

$$(F_3, R_9) = c_4(12) + l(3) + 2s_3(6) + 6s_2(6).$$

Of the ten lines on F_3 which cut l, eight lie on R_9; the other two are the residuals in the planes (l, s_3), (l, s_3') and have but one point on c_4. F_3 contains sixteen other lines which do not lie on R_2 nor on R_9.

4. Now suppose no restriction be put upon the relation between P and π, but that c_4 passes through κ. The conic in every plane π will pass through κ. The surface is now of order four, and l is a double line upon it. The only conic common to F_4 and R_2 is in the plane π when P is at κ'; that in the plane corresponding to κ touches l at κ, but does not lie on R_2. In the plane of (l, s_3'), l is itself part of c_2. The point κ is a triple point on the surface. The R_9 of bisecants breaks up into a cubic cone K_3 having its vertex at κ and having s_3 for a double edge, and an R_6 having l for triple line, c_4 for double curve, s_3' for a triple generator, and s_3 for a simple generator.*

We now have

$$(F_4, R_6) = c_4(8) + s_3(1) + s_3'(3) + l(6) + 6s_2(6).$$

The correspondence (ω, π) is now $(3, 3)$ with six coincidences, which account for twelve lines on F_4, but only six belong to R_6, the other six lying on K_3,

$$(F_4, K_3) = c_4(4) + s_3(2) + 6s_1(6).$$

The surfaces F_4, K_3 furnish a monoidal representation of c_4 when the common vertex is a point on the curve.

If in the correspondence (P, π), κ, (l, s_3) are corresponding elements, nothing new will result, since the line joining Q_4 of c_4 to κ is not a generator of R_2. The surface is not changed except that one of the six bisecants of c_4 mentioned above now passes through the triple point κ.

5. If κ', (l, s_3') are corresponding elements however, the surface reduces to a cubic on which l is a simple line and κ is a double point;

$$(F_3, R_2) = c_4(4) + 2s_3(2); \quad (F_3, R_6) = l(3) + c_4(8) + s_3(1)$$
$$+ s_3'(3) + 3s_2(3), \quad (F_3, K_3) = c_4(4) + s_3(2) + 3s_1(3).$$

F_3 and K_3 furnish a monoidal representation of c_4.

6. If l is a bisecant of c_4, every c_2 must pass through two fixed points. Since it must also pass through P on l, the latter is a factor of every c_2, and the residual is a straight line joining

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* This surface is type 80 in my enumeration of sextic scrolls, *Amer. Jour. of Math.*, vol. 27, p. 101.

the other two points of c_4 in π. The surface is now of order three and l is a double line above it. Since from every point of l two bisecants to c_4 can be drawn, apart from l itself, the surface is a cubic scroll of the first kind, R_3. The R_6 breaks up into a cubic cone, with vertex at κ', and this same R_3.

Finally, if l has three points upon c_4, F becomes the R_2 upon which c_4 lies.

7. If c_4 is of genus one it has no trisecants. As no line can cut more than two conics apart from the two lying on R_2, the surface is of order four and l is a double line upon it. The scroll of bisecants is now R_8 on which l is a double directrix and c_4 a triple curve. The correspondence (π, ω) is now (2, 6) with eight coincidences and 16 lines, hence

$$(F_4, R_8) = c_4(12) + l(4) + 16s_2(16).$$

If one or both points κ, κ' correspond to the plane containing a bisecant passing through them, the conics in these planes break up, but no important changes in the form of the surfaces occur.

8. If l intersects c_4 at κ, R_8 breaks up into an elliptic K_3 and an R_5 having the symbol $l_2^2 + c_4^3$.* F is a cubic on which l is a simple line and κ is a node.

$$(F_3, R_5) = l(2) + c_4(8) + 5s_2(5), \quad (F_3, K_3) = c_4(4) + 5s_1(5).$$

The correspondence (π, ω) is now (2, 3) and the residual lines in the planes of the coincidences belong to K_3. If l intersects c_4 twice the surface reduces to R_2 containing c_4 and l.

9. If c_4 has a node, and no restrictions as to (π, P), the surface is F_4, having l for double line. Two lines through the node cut l and cut c_4 again;

$$(R_8, F_4) = l(4) + c_4(12) + 16s_2(16).$$

If κ (l node) are corresponding elements, the surface is F_3,

$$(F_3, R_8) = l(2) + c_4(12) + 10s_2(10),$$

since the line joining the node to κ and cutting c_4 again counts for three coincidences. Both points κ, κ' in which l cuts K_2 on which c_4 lies cannot give rise to coincidence, because they lie in the same plane κ.

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* This scroll is type B, iv of Schwarz's classification in *Crelle's Journal*, vol. 67, p. 37.

10. If l cuts c_4 in κ, the surface is a nodal cubic, having one node at κ, and another at the node of c_4. R_8 breaks up into K_3 and R_5, the latter having the symbol $l_2 + c_4^2 + 3^2$ (Schwarz's A, vii), the double generation being the line joining the node to the fourth point in the plane containing l. Thus,

$$(F_3, R_5) = l(2) + c_4(8) + g_2(2) + 3s_2(3).$$

If c_4 has a cusp, the second nodal point becomes uniplanar. Further specializations result in quadrics and quadric cones.

CORNELL UNIVERSITY,
January, 1906.

OPERATION GROUPS OF ORDER $p_1^{m_1\mu_1}p_2^{m_2\mu_2}$.

BY PROFESSOR O. E. GLENN.

IT is desired to make certain generalizations concerning the groups of order the product of powers of two primes p_1, p_2, such that $p_1 \equiv 1 (\text{mod } p_2)$, these groups possessing abelian subgroups H_i of type $[\mu_i, \mu_i, \cdots, \mu_i]$ $(i = 1, 2)$. It is possible to specify for these groups those subgroups (here called basic subgroups) from which it is necessary and sufficient that generating operations be selected in order that they may generate the whole group G. This general problem connected with groups of composite order seems to merit more attention than it has thus far received. If

$$H_1 = \{P_1, P_2, \cdots, P_{m_1}\}, \quad H_2 = \{Q_1, Q_2, \cdots, Q_{m_2}\},$$

then the number of operations of order $p_i^{\mu_i}$ in H_i is

$$\sum_{j=0}^{m_i-1} {}_{m_i}C_j[p_i^{\mu_i} - \Phi(p_i^{\mu_i})]^j\,[\Phi(p_i^{\mu_i})]^{m_i-j}$$

$$= [p_i^{\mu_i-1} + \Phi(p_i^{\mu_i})]^{m_i} - p^{m_i(\mu_i-1)},$$

so that the number of cyclical subgroups of order $p_i^{\mu_i}$ in H_i is

$$N_{p_i}^{\mu_i} = \frac{p_i^{m_i(\mu_i-1)}(p_i^{m_i} - 1)}{\Phi(p_i^{\mu})}$$

$$= p_i^{(m_i-1)(\mu_i-1)}(p_i^{m_i-1} + p_i^{m_i-2} + \cdots + p_i + 1).$$

Let us now suppose that $\mu_1 = \mu_2 = 1$. Transformation of the $N^1_{p_1}$ subgroups of H_1 by Q_κ permutes them in sets of p_2 or else leaves them invariant. If ρ_{Q_κ, p_1} is the number left invariant

$$N^1_{p_1} - \rho_{Q_\kappa, p_1} \equiv 0 \pmod{p_2},$$

and when $p_1 \equiv 1 \pmod{p_2}$, $\rho_{Q_\kappa, p_1} \geqq m_1$. It follows that P_λ $(\lambda = 1, 2, \cdots, m_1)$ can be so selected that

$$(1) \qquad Q_1^{-1} P_\lambda Q_1 = P_\lambda^{\alpha^{x_{1\lambda}}}, \quad \alpha^{p_2} \equiv 1 \pmod{p_1}.$$

Now H_1 is self-conjugate * in G hence

$$Q_2^{-1} P_\lambda Q_2 = P_1^{a_{1\lambda}} P_2^{a_{2\lambda}} \cdots P_{m_1}^{a_{m_1\lambda}},$$

and

$$(Q_1 Q_2)^{-1} P_\lambda (Q_1 Q_2) = \prod_{\nu=1}^{m_1} P_\nu^{a_{\nu\lambda} \alpha^{x_{1\lambda}}} = (Q_2 Q_1)^{-1} P_\lambda (Q_2 Q_1)$$
$$= \prod_{\nu=1}^{m_1} P_\nu^{a_{\nu\lambda} \alpha^{x_{1\nu}}}.$$

$$(2) \qquad a_{\nu\lambda}(\alpha^{x_{1\lambda}} - \alpha^{x_{1\nu}}) \equiv 0 \pmod{p_1},$$

and if $x_{1\lambda} \not\equiv x_{1\nu} \pmod{p_2}$ then $a_{\nu\lambda} \equiv 0 \pmod{p_1}$. Suppose on the other hand that some of the $x_{1\lambda}$ are congruent to some of the $x_{1\nu} (\nu \not\equiv \lambda)$, say $x_{1\lambda} \equiv x_{1\nu} (\nu = \lambda + 1, \lambda + 2, \cdots, \lambda + r)$, then $a_{\lambda+i}$, may be zero or not. Assuming the latter, we prove that *all the subgroups in the group* $\{P_\lambda, P_{\lambda+1}, P_{\lambda+2}, \cdots, P_{\lambda+r}\}$ *are permutable with* Q_1, *and the operations* $P_\lambda, P_{\lambda+1}, P_{\lambda+2}, \cdots, P_{\lambda+r}, Q_2$ *form a subgroup of order* $p_1^{r+1} p_2$.

The first statement is obvious since

$$Q_1^{-1} \prod_{i=0}^{r} P_{\lambda+i}^{a_i} Q_1 = \left(\prod_{i=0}^{r} P_{\lambda+i}^{a_i} \right)^{a^{x_{1\lambda}}}.$$

To prove the latter we have

$$Q_2^{-1} P_\lambda Q_2 = P_\lambda^{a_{\lambda\lambda}} P_{\lambda+1}^{a_{\lambda+1,\lambda}} \cdots P_{\lambda+r}^{a_{\lambda+r,\lambda}} = \prod_{i=0}^{r} P_{\lambda+i}^{a_{\lambda+i,\lambda}}.$$

Also

$$(Q_1 Q_2)^{-1} P_{\lambda+i} (Q_1 Q_2) = P_1^{a_{1,\lambda+i} a^{x_{1\lambda}}} P_2^{a_{2,\lambda+i} a^{x_{1\lambda}}} \cdots P_{m_1}^{a_{m_1\lambda+i} a^{x_{1\lambda}}}$$
$$= \prod_{j=1}^{m_1} P_j^{a_{j,\lambda+i} a^{x_{1\lambda}}} = (Q_2 Q_1)^{-1} P_{\lambda+i} (Q_2 Q_1)$$
$$= P_1^{a_{1,\lambda+i} a^{x_{11}}} P_2^{a_{2,\lambda+i} a^{x_{12}}} \cdots P_\lambda^{a_{\lambda,\lambda+i} a^{x_{1\lambda}}} \cdots P_{\lambda+r}^{a_{\lambda+r,\lambda+i} a^{x_{1\lambda}}} \cdots P_{m_1}^{a_{m_1\lambda+i} a^{x_{1m_1}}},$$

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* Burnside, Finite Groups, p. 351, Theorem V.

and since $$x_{1k} \neq x_{1\lambda} \quad (k \neq \lambda, \lambda + 1, \cdots, \lambda + r),$$

$$a_{j, \lambda+i} \equiv 0 \pmod{p_1} \quad (j \neq \lambda, \lambda + 1, \cdots, \lambda + r),$$

so that

$$Q_2^{-1} P_{\lambda+i} Q_2 = P_\lambda^{a_{\lambda, \lambda+i}} P_{\lambda+1}^{a_{\lambda+1, \lambda+i}} \cdots P_{\lambda+r}^{a_{\lambda+r, \lambda+i}} = \prod_{j=\lambda}^{\lambda+r} P_j^{a_{j, \lambda+i}}$$

$$(i = 0, 1, 2, \cdots, r),$$

and the operations P_j, Q_2 form a subgroup (I) of order $p_1^{r+1} p_2$. The number of subgroups of order p_1 in (I) is

$$p_1^r + p_1^{r-1} + \cdots + p_1 + 1,$$

and at least $r + 1$ of them are permutable with Q_2, since $p_1 \equiv 1 \pmod{p_2}$. If these are $\{P'_j\}$ $(j = \lambda, \lambda + 1, \cdots, \lambda + r)$, then, on replacing P_j by P'_j, the series of m_1 independent generators of H_1

$$P_1, P_2, P_3, \cdots, P_{\lambda-1}, P'_\lambda, P'_{\lambda+1}, \cdots, P'_{\lambda+r}, P_{\lambda+r+1}, \cdots, P_{m_1}$$

has the property that each generator is transformed into one of its own powers by both Q_1 and Q_2. Hence, referring back to equation (2), even when $x_{1\lambda} \equiv x_{1\nu} \pmod{p_2}$ we may write $a_{\nu\lambda} \equiv 0 \pmod{p_1}$ $(\nu \neq \lambda)$. It is also obvious that when $\nu = \lambda$, then $a_{\lambda\lambda} \equiv$ some power of α, as $\alpha^{x_{2\lambda}}$. Hence G is defined by the relations

$$P_\lambda^{p_1} = Q_\kappa^{p_2} = 1,\ P_\lambda P_\nu = P_\nu P_\lambda,\ Q_\kappa Q_l = Q_l Q_\kappa,\ Q_\kappa^{-1} P_\lambda Q_\kappa = P_\lambda^{a^{x_{\kappa\lambda}}}$$

$$(\lambda, \nu = 1, 2, \cdots, m_1;\ \kappa, l = 1, 2, \cdots, m_2)$$

$$a^{p_2} \equiv 1 \pmod{p_1}, \quad x_{\kappa\lambda} \equiv 0, 1, \cdots, p_2 - 1 \pmod{p_2}.$$

LEMMA: *There are $m_2 - 1$ of the $x_{\kappa\lambda}$ equal to zero, and no generality is lost by assuming that*

$$x_{11} \equiv x_{21} \equiv x_{31} \equiv \cdots x_{m_2-1, 1} \equiv 0,\ x_{m_2 1} \equiv 1 \pmod{p_2}.$$

Since G is by assumption non-divisible, there is at least one operation, as Q_{m_2}, which transforms P_1 into $P_1^a (a \neq 1)$. Say $Q_{m_2}^{-1} P_1 Q_{m_2} = P_1^a$. If $Q'_{m_2-\tau}$ is any other generator of H_2, by what has been given

$$Q'^{-1}_{m_2-\tau} P_1 Q'_{m_2-\tau} = P_1^{a^{x'_{m_2-\tau, 1}}} = Q_{m_2}^{-x'_{m_2-\tau, 1}} P_1 Q_{m_2}^{x'_{m_2-\tau, 1}},$$

$$\therefore \left(Q'_{m_2-\tau} Q_{m_2}^{-x'_{m_2-\tau,1}} \right)^{-1} P_1 \left(Q'_{m_2-\tau} Q_{m_2}^{-x'_{m_2-\tau,1}} \right) = P_1,$$

$$(= Q_{m_2-\tau}^{-1} P_1 Q_{m_2-\tau} \text{ say}).$$

It follows that $Q_{m_2-\tau}$ $(\tau = 1, 2, \cdots, m_2 - 1)$ is permutable with P_1, so that $x_{11} \equiv x_{21} \cdots \equiv x_{m_2-1,1} \equiv 0$ and $x_{m_2 1} \equiv 1$. By this means all the generators of order p_2 of the group G are fixed and no other invariants $x_{\kappa\lambda}$ can be placed equal to zero without destroying the generality of the defining relations.

The most general form of the generating operations of order p_1 of G will now be derived.

Let

$$\bar{P}_t = P_1^{\alpha_{1t}} P_2^{\alpha_{2t}} \cdots P_{m_1}^{\alpha_{m_1 t}}; \quad \bar{Q}_u = Q_1^{\beta_{1u}} Q_2^{\beta_{2u}} \cdots Q_{m_2}^{\beta_{m_2 u}},$$

and write

$$\bar{Q}_u^{-1} \bar{P}_t \bar{Q}_u = \bar{P}_t^{\alpha^{x_{ut}}}.$$

Under the assumption $\alpha_{it} \not\equiv 0 \pmod{p_1}$ we get

$$(3) \quad \beta_{m_2 u} x_{m_2 i} + \beta_{m_2-1 u} x_{m_2-1 i} + \beta_{m_2-2 u} x_{m_2-2 i} + \cdots + \beta_{1u} x_{1i} \equiv x_{ut} \pmod{p_2} \quad (i = 1, 2, \cdots, m_1),$$

so that for a chosen t (say $t = 1$) and a chosen u (as $u = 1$) we have a set of m_1 linear congruences in the β_{ku} (as the following):

$$(4) \qquad \left. \begin{array}{l} \beta_{m_2 1} \equiv 0, \\ \beta_{m_2-1 1} x_{m_2-1, 2} + \cdots + \beta_{11} x_{12} \equiv 0, \\ \beta_{m_2-1 1} x_{m_2-1, 3} + \cdots + \beta_{11} x_{13} \equiv 0, \\ \cdots\cdots\cdots\cdots\cdots \\ \beta_{m_2-1 1} x_{m_2-1, m_1} + \cdots + \beta_{11} x_{1 m_1} \equiv 0, \end{array} \right\} \pmod{p_2}.$$

If no $\alpha_{i1} \equiv 0$, then assuming $m_1 = m_2$,

$$\begin{vmatrix} x_{m_2-1, 2} & \cdots & x_{12} \\ x_{m_2-1, 3} & \cdots & x_{13} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ x_{m_2-1, m_1} & \cdots & x_{1 m_1} \end{vmatrix} \equiv 0 \pmod{p_2},$$

a relation among the invariants of the group which is not possible *in general.* Hence since we seek *general* generators of the group we must have at least one α_{i1} ($\neq \alpha_{11}$ since $\bar{P}_1$ obviously $= P_1^{\alpha_{11}}$ as a special case) and one of the β_{k1}; viz., β_{m_21} equal to zero. Again if $m_1 = m_2 + \sigma$ (σ = integer), then at least $\sigma + 1$ of the α_{i1} ($\neq \alpha_{11}$) are zero. If $m_1 < m_2$, no α_{i1} is necessarily zero, by virtue of (4). In the same way if $u = 2$, $t = 1$, $\beta_{m_2,2} \equiv 0$, and in general if $u = u(< m_2)t = 1$, $\beta_{m_2u} \equiv 0$; the same condition holding for $m_1 \gtreqless m_2$ as above. When $u = m_2$, $t = 1$ we have

$$\beta_{m_2m_2}x_{m_21} \equiv 1, \text{ (whence } \beta_{m_2m_2} \equiv 1 \pmod{p_2}).$$

$$\left.\begin{array}{l} x_{m_22} + \beta_{m_2-1m_2}x_{m_2-12} + \cdots + \beta_{1m_2}x_{12} \equiv 1, \\ x_{m_23} + \beta_{m_2-1m_2}x_{m_2-13} + \cdots + \beta_{1m_2}x_{13} \equiv 1, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_{m_2m_1} + \beta_{m_2-1m_2}x_{m_2-1m_1} + \cdots + \beta_{1m_2}x_{1m_1} \equiv 1, \end{array}\right\} \pmod{p_2}.$$

And except for $\beta_{m_2m_2} \equiv 1$, the conclusions are the same as above; and the argument may be repeated for $t = 2, 3, \ldots, m_1$ in equation (3). The general result is the following:

THEOREM: *The necessary and sufficient conditions that a given set of m_i operations of order p_i ($i = 1, 2$) of G shall generate G are* (1) *that they be independent,* (2) *if $m_1 = m_2 + \sigma$ then $\overline{P_t}$ must be selected from a certain basic subgroup of H_1, of order $\leqq p_1^{m_1-\sigma-1}$, in which of course P_t occurs; $\overline{Q_u}(u<m_2)$ is likewise an operation of a subgroup of order $p_2^{m_2-1}$, containing Q_u, viz., of $\{Q_1, Q_2, \cdots, Q_{m_2-1}\}$. It immediately follows since the $x_{\kappa\lambda}$ are invariants of the group that these basic subgroups must be of order p_1 (one $a_{it} \neq 0$) and coincide with the $\{P_t\}$. The number of groups isomorphic to a given type (given set of invariants $x_{k\lambda}$) is equal to the number of distinct types obtained by transforming G by all $m_1!$ permutations of the basic subgroups.*

In particular if $m_2 = 1$, so that $m_1 = m_2 + m_1 - 1$ the basic subgroups are of order $p_1^{m_1-m_1+1} = p_1$. In other words $\overline{P}_t = P_t$ and $\overline{Q_1} = Q_1^{\beta_{11}}$, and all the types isomorphic to a given type are obtainable by permuting the m_1 subgroups $\{P_t\}$ in the $m_1!$ possible ways. The number of non-isomorphic types has been determined for this case in the form*

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*BULLETIN, vol. 11, No. 6, p. 318.

$$N = \frac{1}{m_1}\left[\sum_{\sigma=0}^{(m_1-1)(p_2-2)} P(0, 1, \cdots, p_2 - 2)^{m_1-1}\sigma + \psi\right],$$

where $P(0, 1, \cdots, p_2 - 2)^{m_1-1}\sigma$ stands for the number of partitions of σ in $(m_1 - 1)$'s by the numbers $0, 1, \cdots, p_2 - 2$; and ψ is a determinate function of p_2 and m_1.

SPRINGFIELD, MO.,
December, 1905.

A DEFINITION OF QUATERNIONS BY INDEPENDENT POSTULATES.*

BY MISS E. L. CARSTENS.

(Read before the American Mathematical Society, February 24, 1906.)

§ 1. *Quaternions with respect to a Domain D.*†

THE usual theory relates to quaternions $a_1 + a_2 i + a_3 j + a_4 k$ in which the coefficients a_i range independently over all real numbers or else over all complex numbers, and the units have the following multiplication table:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

These conditions give the real quaternion system and the octonion system.‡ As an obvious generalization, the coefficients may range independently over all the elements of any domain D.

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*See Dickson, "On hypercomplex number systems," *Transactions Amer. Math. Society*, vol. 6 (1905).

† A domain consists of any class of elements.

‡ Octonions may be considered as quaternions with complex coefficients.

§ 2. *The Postulates.*

A set of four ordered elements $a = [a_1, a_2, a_3, a_4]$ of D will be called a quaternion a. The symbol $a = [a_1, a_2, a_3, a_4]$ employed is purely positional, without functional connotation. Its definition implies that $a = b$ if and only if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, $a_4 = b_4$.

Postulate I. If a and b are any two quaternions, then $s = [a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4]$ is a quaternion.

Definition. Addition of quaternions is defined by $a \oplus b = s$.

Postulate II. $0 = [0, 0, 0, 0]$ is a quaternion.

Postulate III. If 0 is a quaternion, then to any quaternion a corresponds a quaternion a' such that $a \oplus a' = 0$.

Theorem 1. Quaternions form a commutative group under addition.

Postulate IV. a and b being any two quaternions, then $a \oplus b = p = [p_1, p_2, p_3, p_4]$ is a quaternion, where

$$p_1 = a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4, \quad p_3 = a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2$$

$$p_2 = a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3, \quad p_4 = a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1,$$

if the p_i's are in D.

Definition. The product of two quaternions is defined by $a \otimes b = p$.

Theorem 2. Multiplication is not commutative.

Theorem 3. Multiplication is distributive (right and left).

Theorem 4. Multiplication is associative.

To make quaternions four dimensional we add a fifth postulate:

Postulate V. If $\tau_1, \tau_2, \tau_3, \tau_4$ are elements of D such that $\tau_1 a_1 + \tau_2 a_2 + \tau_3 a_3 + \tau_4 a_4 = 0$ for every quaternion a, then $\tau_1 = 0$, $\tau_2 = 0$, $\tau_3 = 0$, $\tau_4 = 0$.

Theorem 5. There exist four quaternions $\epsilon_i = [a_{i1}, a_{i2}, a_{i3}, a_{i4}]$ such that $|a_{ij}| \neq 0$.

§ 3. *Identification with Ordinary Quaternions.*

The quaternion system as thus defined is holoedrically isomorphic with the quaternions of Hamilton, the coefficients belonging to the same domain D.

The quaternions $e_1 = [1, 0, 0, 0]$, $e_2 = [0, 1, 0, 0]$, $e_3 = [0, 0, 1, 0]$, $e_4 = [0, 0, 0, 1]$ form a four dimensional system since

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \neq 0.$$

By postulate IV and the definition of multiplication, these quaternions e have the multiplication table

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$-e_4$	$-e_1$	e_2
e_4	e_4	e_3	$-e_2$	$-e_1$

which, apart from symbolism, is the same as the table of § 1.

§ 4. *On the Independence of the Postulates.*

If D is a domain admitting addition and subtraction, postulates II and III are redundant.

Aside from this case, postulates I–V are independent as shown by the following systems:

(I) Elements 0, [± 1, 0, 0, 0], [0, ± 1, 0, 0], [0, 0, ± 1, 0], [0, 0, 0, ± 1].

(II) D is the domain of positive integers.

(III) Set (II) with 0 added.

(IV) D is the domain of complex numbers, the a_i being pure imaginaries.

(V) a_1 arbitrary; other a's $= 0$.

THE UNIVERSITY OF COLORADO,
BOULDER, COLO.,
February, 1906.

NOTE ON THE HEINE–BOREL THEOREM.

BY MR. N. J. LENNES.

(Read before the Chicago Section of the American Mathematical Society, December 29, 1905.)

IN this note is presented a generalized form of the Heine-Borel theorem together with a corollary which is of immediate application in many theorems in metrical analysis. The present form of the theorem is the result of an effort to understand the meaning of the properties of boundedness and unboundedness of sets of points (numbers) in non-metrical analysis situs. It appears that for many purposes the property of boundedness when applied to a closed set may be replaced by the property that the set shall contain its limit points at infinity, *i. e.*, the set shall be closed even if it is unbounded. In non-metrical analysis, therefore, the chief distinction between a closed bounded set and an unbounded set not containing its limit points at infinity seems to be that the latter is necessarily not closed.

The theorem is stated for the case of three dimensions and the language of geometry is used exclusively. In view of the one-to-one correspondence of the set of all points in a three-space and the set of all triples of real numbers, the reader may, if he wishes, regard the geometric language as a notation for a three dimensional number manifold.

§ 1. *Definitions and Preliminary Notions.*

The word region is used to denote any set of points whatever. Two half-lines proceeding from the same point O and not lying in the same line form an angle. (The half-line contains the point O from which it proceeds.) We assume that an angle separates the remaining points of the plane in which it lies into two unique sets, an interior and an exterior set. If the half-lines a, b, c, no two of which lie in the same line and all three of which do not lie in the same plane, proceed from the same point O then the three angles formed by these half-lines together with the interior points of these angles form a trihedron $Oabc$. If A, B, C, are points of the respective half-lines a, b, c, then the four triangles OAB, OBC, OCA, and ABC, together with their interior points, form a tetrahedron. Such tetrahedron we shall speak of as associated with the trihedron $Oabc$. We assume that a trihedron and also a tetrahedron separates space into two unique sets, and hence that these figures have a definite interior.

A finite point P' is a limit point of a set $[P]$ of points if there are points of $[P]$ other than P' within every tetrahedron of which P' is an interior point. If a set $[P]$ of a half-line is unbounded then the set $[P]$ is said to have a limit point ∞ on the half-line. We will regard every half-line as having one point at infinity. A point ∞ on a half-line l proceeding from a point O is said to be a limit point of a set $[P]$ if for every trihedron $Oabc$ containing the points of l as interior points, and for every associated tetrahedron $OABC$ there are points of $[P]$ other than ∞ within $Oabc$ and exterior to $OABC$. A point is within a region R if it is not a limit point of points not of the region. This definition applies to points at infinity, *i. e.*, a point ∞ lies within a region R if there exists a trihedron $Oabc$ and an associated tetrahedron $OABC$ such that ∞ is an interior point of $Oabc$ and further such that every point, other than ∞, within $Oabc$ and exterior to $OABC$ is a point of R. A closed set contains all its limit points. This also applies to points at infinity.

§ 2. *The Generalized Heine-Borel Theorem.*

THEOREM: *If $[P]$ is a closed set of points and if $[R]$ is a set of regions such that every point of $[P]$ is an interior point of at least one region of the set $[R]$ then there exists a finite subset $R_1, R_2, \cdots, R_n$, of regions of the set $[R]$ such that every point of $[P]$ lies within at least one region of the set $R_1, R_2, \cdots, R_n$.*

Proof: Consider any trihedron $Oabc$. Denote by $[P_a]$ the points of $[P]$ which lie on the half-line a. We show first that there is a finite subset of $[R]$, $R_1, R_2, \cdots, R_{n_1}$, such that every point of $[P_a]$ lies within at least one of the regions $R_1, R_2, \cdots, R_{n_1}$. Since the set of all points common to two closed sets is a closed set it follows that the set $[P_a]$ is closed. If the point ∞ of a is a point of $[P_a]$ it follows that there is a region of the set [R] which contains some infinite segment $K\infty$ of a. It remains to show that all points of $[P_a]$ on OK are contained in some finite set of regions of $[R]$. Beginning at O denote by $[P_a']$ the set of all points of $[P_a]$ on OK which are contained within some finite set of regions. This set of points has a least upper bound $\bar{B}[P_a']$, which is a point of

$[P_a]$ since $[P_a]$ is closed. By hypothesis $\bar{B}\ [P_a']$ lies within one region of $[R]$. Then either $\bar{B}[P_a']$ is a limit point of points of $[P_a']$ which are on $\bar{B}[P_a']K$, in which case $\bar{B}[P_a']$ fails to be an upper bound of $[P_a']$, or there is a definite next point of $[P_a]$ on $\bar{B}[P_a']K$ and lying within a region of $[R]$. Hence $B[P_a']$ cannot be an upper bound as just specified. Hence the points of $[P_a]$ on OK which lie within a finite set of regions of the set $[R]$ has no upper bound and therefore there exists a set of regions $R_1, \cdots, R_{n_1}$ of the set $[R]$ such that every point of $[P_a]$ lies within at least one of them.

Obviously there are half-lines k proceeding from O and lying within the angle (a, b) such that every point of $[P]$ which lies on or within the angle (a, b) lies within at least one of the regions $R_1, \cdots, R_{n_1}$. If there does not exist a finite set of regions of the set $[R]$ such that every point of $[P]$ which lies on or within the angle (a, b) lies within at least one of them, then let k' be the bound of all half-lines proceeding from O such that the points of $[P]$ which lie on or within the angle formed by them and a lie within such finite set of regions. By the above proof all points of $[P]$ which lie on k' lie within a finite set of regions of the set $[R]$. Hence k' cannot be a bound as specified, and every point of $[P]$ which lie on or within the angle (a, b) lies within at least one of a certain finite set $R_1, R_2, \cdots, R_{n_2}$ of regions of the set $[R]$. In the same manner we may show that there exists a subset of $[R]$, $R_1, R_2, \cdots, R_{n_3}$ such that every point of $[P]$ which lies on or within the trihedron $Oabc$ lies with in at least one region of the set $R_1, \cdots, R_{n_3}$. But the three planes determined by the faces of the trihedron $Oabc$ separate the three-space into eight trihedrons of the type $Oabc$. whence the theorem follows for the whole set $[P]$.

This theorem has many immediate corollaries, of which I instance the two following.

1. *Every infinite set of points has at least one limit point which may be finite or infinite.*

2. *If* $z = f(x, y)$ *and if* (a, b) *is a limit point of the set* (x, y) *on which* $f(x, y)$ *is defined then* $z = f(x, y)$ *has some value approached as* (x, y) *approach* (a, b).

Proof: If the theorem fails to hold, then about every point of the line $x = a, y = b$ there is some region within which there is no point of the graph of the function $z = f(x, y)$. By the theorem of this note there is a finite subset of these regions and hence a cylinder enclosing the line $x = a$, $y = b$ within which there is no point of the graph of the function. Hence for values of x and y within this cylinder there are no values of the function, which is contrary to the hypothesis that the function is defined on a set which has (a, b) as a limit point.

§ 3. *A Theorem of Continuity.*

THEOREM: *If $[P]$ is a closed bounded set of points and if $[R]$ is a set of regions such that every point of $[P]$ is an interior point of at least one region of the set then there exists a number d such that a cube whose edge is d and whose center is any point whatever of the set $[P]$ will lie entirely within one of the regions of the set $[R]$.*

Proof: About every point of $[P]$ construct a cube c lying within some region of the set $[R]$. Let the sides of the cubes be parallel to the coördinate planes of a fixed rectangular system. By the Heine-Borel theorem there is a finite subset $c_1, \cdots c_n$ of these cubes such that every point of $[P]$ lies within at least one of them. Obviously, that part of the surface of any cube of $c_1, \cdots, c_n$ which is exterior to all other cubes of the set is made up of a finite number of rectangles. Denote the set of all such rectangles obtained from $c_1, \cdots, c_n$ by $[r]$. No such rectangle contains a point of $[P]$ or is a limit point of $[P]$. Hence there exists a positive number d' which is less than the distance from any point of $[P]$ to any point of $[r]$. Let d' be a positive number less than the distance from any side s of a cube of $c_1, \cdots, c_n$ to any parallel side (the plane of which does not contain s). Let d''' be the smaller of these two numbers then $d = \frac{1}{2}d'''$ is the required number.

This theorem appears to have been stated in essentially the above form by several persons during the last year. Professor Bolza used it for a set of points in a plane as early as the spring of 1905, and it is possible that the present statement of it is partly due to his suggestion. Mr. Wedderburn has also used the theorem in this form for linear sets. So far as I know it has not been published before now. Dr. Veblen suggested that this form of the theorem be called "the theorem of uniformity." The following is an immediate corollary: "*A function which is continuous over an interval is uniformly continuous over that interval.*"

CHICAGO,
February 22, 1906.

FUNCTIONS OF REAL VARIABLES.

Leçons sur les Fonctions de Variables réelles et les Développements en Séries de Polynomes. Professées a l'École normale supérieure par ÉMILE BOREL, et rédigées par MAURICE FRÉCHET. Avec des notes par PAUL PAINLEVÉ et HENRI LEBESGUE. Paris, Gauthier-Villars, 1905. viii + 160 pp.

THE theory of functions has for several years been distinctly the fashionable thing in mathematical circles in Paris, and the publication of Borel's collection of monographs on the subject, of which the volume now under review is the eighth, aims to gain devotees elsewhere. For we take the chief purpose of this series of publications to be the encouragement of research by the presentation in connected form of the more recent results in the theory of functions, each volume confining itself to some limited domain of investigation. So the present volume confines itself in the main to a connected exposition of the present state of the theory of series of polynomials of a real variable, connected not merely in itself, but also with the older theory.

This volume, like its predecessors, is characterized by great precision of statement, rigor of proof, and elegance of treatment throughout. If a single word were desired to describe the whole, we could think of no better than "modern." The book is fairly alive with the modern spirit. As an instance we may note the fact that the so-called Heine-Borel theorem, which the reviewer believes should play a fundamental rôle in every modern treatment of the theory of functions of real variables, is introduced at the outset (page 9) and is used extensively throughout the volume, greatly simplifying many proofs and adding largely to the general elegance of the exposition.

As has already been said, one of the characteristics of the volume is precision of statement. It is strange, therefore, that one of the few criticisms that we have to make should be for a lack of precision. On page 6 the author speaks of "the segment (0, 1), which we will call the *fundamental interval.*" We are not told however that the end points are to be included. On the following page it becomes evident that 0 and 1 are points of the fundamental interval; and in fact throughout the volume the symbol (a, b) seems to be used for the set of points x satisfying the condition $a \leqq x \leqq b$ (in this review we will so understand the symbol), but this is nowhere stated explicitly, and the occurrence of the phrase "the interval (a, b), extremities included" (page 27, *e. g.*) seems to imply that the last two qualifying words are necessary. This we believe is a lack of precision in notation. It would be better surely to introduce at the outset unique symbols to denote intervals with end points included, and with one or both excluded.*

As may be inferred from the above, the first of the five chapters that make up the body of the volume concerns itself with the theory of point sets. Many of the definitions and theorems here given have of course been known for some time. And yet ideas of recent date are introduced almost at once and with telling effect in the simplification of older proofs. Thus the theorem of Cantor-Bendixson that any closed set can be resolved into a perfect set and an enumerable set, which was originally proved by the aid of transfinite numbers, receives a simple elementary proof by the use of Lindelöf's notion of a point of condensation.

The notion of the measure of a point set (in the Borel sense) receives extended treatment. It is fundamental in much of what follows. What the author calls the fundamental theorem of measure, viz., that the points of an enumerable set of intervals whose total length is less than unity cannot exhaust all the points of an interval of length one, follows almost immediately from the Heine-Borel theorem, already referred to. The general notion of measure is introduced in practically the same way as in the author's earlier volume, Leçons sur la théorie des fonctions (1898). This method of treatment is in so far unsatisfactory that it does not furnish a general means of determining whether or not a given point set is measurable. However, it is possible to show that every point set which is explicitly defined by certain very general operations is measurable. The author, in fact, goes further. He claims that any set which can be explicitly constructed (que nous pourrons *effectivement* former) must be definable in terms of the operations referred to, and hence that every set that we can explicitly define is measurable.

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* Peano has used the symbols $\overline{ab}$, $\overline{ab}$, $\overline{ab}$, $\overline{ab}$, to denote respectively all points x satisfying the conditions $a \leqq x \leqq b$, $a \leqq x < b$, $a < x \leqq b$, $a < x < b$. (Cf. Lezioni di analisi infinitesimale, vol. 1 (1893), p. 9.) More recently Pierpont, Theory of functions of real variables, vol. 1, p. 119, has suggested the symbols (a, b), (a, b^*), (a^*, b), (a^*, b^*) to denote the same sets of points. Moreover, the two words "interval" and "segment" may well be used to distinguish between inclusion and exclusion of end points, the former, say, being used when the end points are included, the latter if one or both of them are excluded. Cf. Veblen, "On the Heine-Borel theorem," BULLETIN, vol. 9 (1904), p. 487.

Continuity is the subject of the second chapter. The definitions are the usual ones. Some of the fundamental theorems receive more or less obvious generalizations. The theorem, for example, that a function which is continuous in an interval (a, b) is uniformly continuous in the interval, is stated more generally as follows, $\omega(x)$ denoting the oscillation at the point x:

If in the interval (a, b) *we have* $\omega(x) \leqq c$, *we can associate with every positive number* ϵ *a positive number* η, *such that the inequality* $|\xi_1 - \xi_2| < \eta$ *implies, for any pair of points* ξ_1, ξ_2 *of* (a, b), *the inequality* $|f(\xi_1) - f(\xi_2)| < c + \epsilon$.*

The proof by means of the Heine-Borel theorem is very simple. The chapter closes with an elegant discussion of Lebesgue's generalization of the notion of integral.

Chapter III. deals with series of real functions, the bulk of the chapter being devoted to a very clear discussion of the conditions under which a convergent series of continuous functions defines a continuous function. The solution of the problem of deriving the necessary and sufficient condition herefor is due to Arzelà. The condition in question, for which Arzelà introduced the term "uniform convergence by segments," is called "quasi-uniform convergence" by M. Borel. Townsend † has used Moore's term "subuniform convergence" as an equivalent, in his recent article on Arzelà's condition; it would seem to be the most convenient. The definition is stated as follows, $r_n(x)$ denoting the remainder after n terms:

A series is said to converge subuniformly in (a, b), *if* 1) *the series converges in* (a, b); 2) *it is possible to associate with every* ϵ, *and every* N, *a finite* $N' > N$, *such that for every* x *in* (a, b) *there exists an integer* n_x *lying between* N *and* N' *such that* $|r_{n_x}(x)| < \epsilon$.

Arzelà's theorem states *that the necessary and sufficient condition that a series whose terms are continuous in* (a, b) *shall define a continuous function in that interval, is that the series shall converge subuniformly in the interval.* This theorem was first given by Arzelà in 1884, but the proof was attacked on the score of rigor. His later proofs (1899 and 1902) which he claims are sound are complicated. Under these conditions one must needs admire the extreme simplicity of the proof given by Borel in this volume. It occupies barely two pages.*

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*In the original b is used where we have used c, thus introducing ambiguity in the use of b.

† BULLETIN, vol. 12 (1905), p. 17. Cf. also Moore, BULLETIN, vol. 7 (1901), p. 257.

The point of departure in the next (fourth) chapter, which deals with the representation of continuous functions by series of polynomials, is naturally the theorem of Weierstrass to the effect that every continuous function can be so represented. Several proofs of this theorem are given, including the original of Weierstrass, that of Volterra, and the purely elementary ones of Runge, Mittag-Leffler and Lebesgue. We find also the extension to functions of several variables, and Borel's theorem on the representation of a continuous function, which has continuous derivatives of every order in the interval $(-1, 1)$ as the sum of a Fourier series and a power series in x. The last eighteen pages of the chapter are devoted to the application of the foregoing theory to the theory of interpolation. A critical examination of Lagrange's formula shows that it can not be relied upon to give an approximation to the given function which increases as the number of known values increases. It is possible, however, *to define once for all polynomials* $P_{p,q}(x)$ *such that any function* $f(x)$, *continuous in* $(0, 1)$, *can be represented by the series, uniformly convergent in* $(0, 1)$,

$$f(x) = \Pi_1 + (\Pi_2 - \Pi_1) + \cdots + (\Pi_n - \Pi_{n-1}) + \cdots,$$

where

$$\Pi_q = \sum_{p=0}^{p=q} f\left(\frac{p}{q}\right) P_{p,q}(x).$$

This is the general formula of interpolation which Borel presented in the summer of 1904 to the Third International Congress of Mathematicians. His procedure is beautifully simple and is briefly as follows: †

Let $\phi_{p,q}(x)$ be a continuous function which in the interval $(0, 1)$ coincides with the function represented by the broken line $OMPNA$ in the figure where P is the point $(p/q, 1)$. It is then clear that if x_1 is any point of the interval $((p-1)/q, p/q)$ and if $\phi_{p,q}(x_1) = \theta$, then $\phi_{p-1,q}(x_1) = 1 - \theta$, and all the other

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* We may refer the reader who is interested in the comparison to the account of Arzela's work in the paper by Townsend already mentioned, which also contains several interesting examples illustrative of the theory.

† Borel's treatment is throughout purely analytic. We have given it this geometric phrasing merely for the sake of greater brevity of statement.

$\phi_{i,q}(x_1)$ vanish. Now, if q be taken so large that the inequality $|x_1 - x_2| < 1/q$ insures the inequality $|f(x_1) - f(x_2)| < \epsilon$ throughout the interval (0, 1), then it follows immediately that

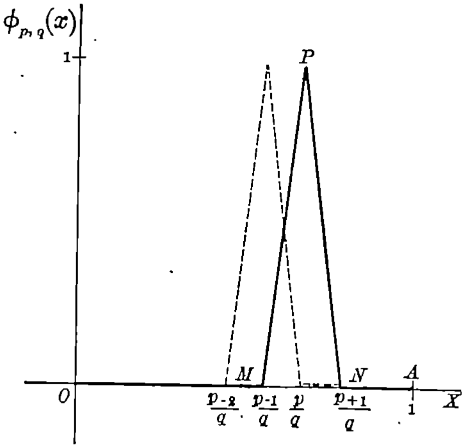
$$\left|\sum_{p=0}^{p=q} f\left(\frac{p}{q}\right)\phi_{p,q}(x) - f(x)\right| < \epsilon$$

throughout (0, 1). It is then only necessary to take for $P_{p,q}(x)$ a polynomial such that the inequality

$$|\phi_{p,q}(x) - P_{p,q}(x)| < 1/q^2$$

holds in (0, 1), to obtain Borel's theorem.

This result is theoretically of much interest. But the reviewer can not agree with M. Borel, when he says in the preface that



this formula is "susceptible of almost immediate application to practical problems of great variety." Any practical application must depend on the explicit calculation of the polynomials $P_{p,q}(x)$, and the nature of the function $\phi_{p,q}(x)$ would seem to require the degree of these polynomials to be very high,* too

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* A method for calculating these polynomials has recently been published (M. Potron, "Sur une formule générale d'interpolation," *Bulletin de la Société mathématique de France*, vol. 34 (1906), p. 52) which makes the degree of $P_{p,q}(x)$ equal to $4q^6$. This number may well be capable of reduction by the use of another method for the calculation; but it is hardly open to question that the polynomial of *lowest* degree satisfying the conditions will still be of little value for practical purposes.

high in any case to be of practical value. In this connection the following modification of Borel's procedure may be of interest. His argument will hold word for word if his function $\phi_{p,q}(x)$ be replaced by any continuous function $\psi_{p,q}(x)$, which satisfies the following conditions: (1) it shall vanish for all points of (0, 1), except for points of the segment $[(p-1)/q, (p+1)/q]$, and (2) for all values of x in the interval $[(p-1)/q, p/q]$ it shall satisfy the relation

$$\psi_{p,q}(x) + \psi_{p,q}(x + 1/q) = 1.$$

The function $\phi_{p,q}(x)$ is theoretically the simplest satisfying these conditions, but it is quite possible that it is not the most convenient for the purpose of representation by means of a polynomial. It would be interesting to determine, if possible, the polynomial $Q_{p,q}(x)$ of lowest degree satisfying the following conditions:

$$1)\ |Q_{p,q}(x)| < \frac{1}{q^2} \text{ if } 0 \leqq x \leqq \frac{p-1}{q} \text{ or } \frac{p+1}{q} \leqq x \leqq 1.$$

$$2)\ |Q_{p,q}(x) + Q_{p,q}(x + \frac{1}{q}) - 1| < \frac{1}{q^2}, \text{ if } \frac{p-1}{q} \leqq x \leqq \frac{p}{q}.$$

This suggests at once a possible generalization of the problem of approximation usually associated with the name of Tchebicheff, where the given function is replaced by an arbitrary function which is required merely to satisfy certain functional relations. The ordinary Tchebicheff problem, *i. e.*, to find that polynomial of given degree which approximates most closely to a given continuous function in a given interval, is given a masterly exposition * in the concluding pages of the chapter.

The eight pages of the fifth and last chapter give a brief introduction to the methods and results recently developed by Baire concerning the representation of discontinuous functions by series of polynomials.

In addition to the five chapters of M. Borel's monograph the volume contains three notes; two short ones by Lebesgue and Borel on "A proof of a theorem by M. Baire" and "On the existence of functions of any class whatever" respectively, and one of 48 pages by Painlevé on "The development of analytic functions." The latter will perhaps be to some the most interesting part of the book, as in it M. Painlevé gives for the first time a connected development of the results which he has recently published in the *Comptes Rendus* concerning the expansion of analytic functions in series that are valid in the whole complex plane with the exception only of certain half-lines, *i. e.*, in the whole "star of holomorphism." But this review is already too long; we can not go into details.

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* On page 87, there is an error of detail; η must be chosen so that $|Q(x)|$ is not only less than $\mu - \epsilon$, but also less than $\mu - \mu'$.

The following are the only typographical errors we have noticed; page 7, line 2, for F read P, and line 8, for M read N; page 16, line 5 from bottom, for E_1 read E_2, and line 4 from bottom, for E_2 read E_1; page 33, line 9, after "limites" insert "inférieures et"; page 58, line 8 in the formula replace y_i by $(y_i - y_{i-1})$; page 60 in the expansion for $\sqrt{x^2}$ the last factors in the numerators of the coefficients of the second and third terms should be removed; page 80, the equations defining [illegible] are not correct when $p = 0$; page 82, in the last sen[illegible] the footnote for Kircherberger read Kirchberger; page [illegible], line 15 from bottom, for A read A'.

J. W. YOUNG.

PRINCETON UNIVERSITY,
March 23, 1906.

SHORTER NOTICES.

Advanced Algebra. By HERBERT E. HAWKES. Ginn and Company, 1905. 285 pp.

IN writing an Advanced Algebra, in the current acceptation of the phrase, two courses are fairly open to an author. He may lay his foundations deep and build on them with unflinching rigor, bringing teacher and pupil into intimate touch with some of the epoch-making researches of the past fifty years. From the standpoint of pure science, this is, of course, admirable; but the meat which it supplies is only for strong men. Foundation-laying is for the hardiest of frontiersmen; and the methodical account of its severely logical steps makes demands upon the average freshman which are far beyond his power to meet.

The second course consists in a frank and explicit assumption of the postulates, as they become necessary in the order of thought, and a frequent appeal to intuition to bridge over awkward chasms. The words "frank and explicit" are used advisedly, in order to differentiate this second course from a vicious middle one, which has often characterized our text books, namely that of making a show of thoroughness without furnishing the genuine article. Nothing is more fatal to deep and fruitful mathematical thinking than such pseudo-rigor.

The book before us follows very conscientiously the second of the courses noted above. The assumptions, so far from being veiled, appear with refreshing boldness as captions and in italics, and the pages ring true. Incidentally it may be remarked that the clear-cut "euclidean" mode of presentation greatly enhances the value of all the books of this Yale series. Every important discussion has a definite outcome and issues in a clearly stated and italicized proposition.

The author also shows courage in his omissions. He leaves out a number of things which for years have been held to belong to the orthodox canon. We look in vain for cube root, the familiar but forbidding proofs of the generalized binomial expansion, convergency, exponential and logarithmic series and the multiplication theorem for determinants. Derivatives, multiple roots and Sturm's theorem have also disappeared. Many thoughtful teachers will miss some of these things and find the book too meager, but the author would rightly rejoin, in the spirit of his preface, that they require the calculus or other portions of mathematics for their complete comprehension. At any rate it must be admitted that he has made good use of the space saved by these omissions, in devoting it to an unusually extended and interesting treatment of graphical representation. If one is seeking for a fine illustration of the utility of graphs in picturing a highly abstract bit of elementary analysis, he need only turn to the figure on page 105, introduced to illustrate the behavior of the general quadratic when the coefficient of x^2 is infinitesimal.

The book begins with a concise account of the fundamental operations and their laws, then proceeds with a rapid review of the usual topics as far as quadratics. Next follows a chapter on mathematical induction, a method which is immediately applied to the derivation of the binomial theorem in the case of positive integral exponents. From this point on we find the customary material of a manual of the kind, with the omissions already referred to. In nearly every chapter, however, the subject matter has been refashioned, clarified and enlivened. If, in the modesty of the author's programme, scholarship may not have its perfect work, yet intimations of its existence are everywhere apparent. Above all else he rarely loses the viewpoint of the teacher or forgets the fact that he is forging a classroom tool.

While in this he has been exceptionally successful, his work is not without its faults. There are errors, some minor and others serious and misleading. One of the characteristics of the book is the repeated use of mathematical induction, yet in the first case in which this method is called into service (page 129) it is abandoned at the end for mere inference. A second attempt to use it (page 177) fails badly because of errors in subscripts and the repeated occurrence of the words "negative roots"— an error which is certainly not typographical. The proposition that imaginary roots enter in pairs is laboriously proved, when eight or ten lines would suffice for a satisfactory demonstration. On page 167 the converse of the corollary should have been stated and proved. While the discussion of the generalized binomial theorem has been perhaps wisely omitted, attention should have been called to the fact that it holds for negative and fractional exponents, and examples furnished to illustrate its use in such cases. In the chapter on determinants (page 221) A_1, A_2, etc., are not "minors," as the author has defined this term, but the old-fashioned cofactors, and this error vitiates the proof. The value of the book, in the opinion of the reviewer, would have been increased by the addition of a brief chapter on the "gradients" of curves, immediately after that on graphs. An experience of several years shows that a very few pages would suffice—and few portions of algebra interest students more. An answer book has been compiled but its value is impaired by the errors which it contains.

But such faults as have been indicated can be easily corrected in a subsequent impression and do not invalidate the conclusion that the book under consideration is admirably adapted to the field which it undertakes to cover. Alike in its theory and in its practice it furnishes fresh, rich and wholesome material for the advanced course in fitting schools and first year work in our colleges.

GEORGE D. OLDS.

Opere matematiche di FRANCESCO BRIOSCHI. Publicate per cura del comitato per le onoranze a Francesco Brioschi (G. Ascoli, E. Beltrami, G. Colombo, L. Cremona, G. Negri, G. Schiaparelli). Volumes 1, 2, 3. Milan, U. Hoepli, 1901, 1902, 1904. xi + 416, viii + 456, x + 435 pp.

BRIOSCHI's life covered seventy-three years (1824–97), the last fifty filled with mathematical labor as teacher, writer, and editor. Taking his doctorate at Pavia in 1845, he became professor of applied mathematics in that university in 1852, and in 1862 was commissioned by the government to found and organize the Istituto tecnico superiore at Milan. There he filled until his death the chairs of hydraulics and analysis, and the office of director. Together with Tortolini he founded in 1858 the *Annali di matematica pura ed applicata* (the immediate successor of the eight volumes of *Annali di scienze matematiche e fisiche*), and carried on independently the enterprise, so significant for the revival of mathematics in Italy, in a second series of twenty-six volumes, 1868–1897. Of his pupils at Pavia the most notable are Cremona, Beltrami, and Casorati. For an account of his labors for the Italian government and a careful analysis of his writings, one should read the excellent biography by Noether in the *Mathematische Annalen*, volume 50 (1898).

The present publication is a part (and the principal part we should esteem it) of the plan to provide a worthy memorial for this great scholar and educator. Scattered in five or more journals, the memoirs embodying Brioschi's extraordinary activity as commentator, critic and creator of modern mathematics had produced their first effect, that on his contemporaries. Now, collected, they become part of the accessible treasure of the scientist and historian. The plan is apparently to collect not his well-known books on determinants and mechanics of rigid systems, but only the shorter papers, from journals and the proceedings of learned societies.

So far there are republished 144 titles. Of these a rough enumeration shows 57 on algebraic invariants, 14 on modern geometry, 14 on differential geometry and the calculus of variations, 19 on differential equations, 21 on the theory of functions and higher analysis, 7 on history, bibliography and biography, 10 on physics or mechanics, and 2 miscellaneous. A more extended résumé is not our purpose here, in view of Noether's paper cited above. This collection reminds one of Cayley's works in its magnitude, but of Clifford's in its large number of brief, pointed papers on topics of general mathematical interest.

The arrangement is by sets, the articles from any one serial appearing together in chronological order. The monograph on the theory of invariants of binary forms is at the close of volume 1, those on the solution of the quintic and sextic are scattered through all of these volumes. Each has a table of contents and a well-executed index. Volume 1 contains the well-known portrait, and the concluding (fifth) volume is to furnish a classified index and a compendious life.

H. S. WHITE.

Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August, 1904. Herausgegeben von dem Schriftführer des Kongresses, DR. A. KRAZER. Leipzig, Teubner, 1905. Royal 8vo, x + 755 pages.

A CONGRESS that can attract four hundred members and hold them for a week, half of them coming from countries outside Germany, must show in its proceedings something worth the study of philosophers, still more that demands the attention of progressive mathematicians. Numbers count for little, but the names of the participants prove the representative and international character of this great gathering. It is gratifying to see Russia occupying second place in number of delegates, and our own country contesting with Denmark the fifth place.

The secretary's report contains the address of Professor Weber, of Strassburg, in opening the Congress, which, it will be remembered, was designed in part as a commemoration of the centenary of the birth of C. G. J. Jacobi. Then follows the address of Professor Königsberger on Jacobi's life and work — dignified, scholarly, full of particulars, but free from technicalities, in short the finished production of the one man who was qualified by his biographical studies and oratorical ability for this onerous task. Next come the four lectures delivered in the general sessions, and these will doubtless prove, for most readers, the most profitable part of this voluminous record. Their titles are as follows: P. Painlevé, "The modern problem of integration of differential equations"; A. G. Greenhill, "The mathematical theory of the top, considered historically"; C. Segre, "The geometry of the present day and its connections with analysis"; W. Wirtinger, "Riemann's lectures upon the hypergeometric series and their significance." Of these the first and third belong to that rare and valuable class of mathematical literature which appears in some prefaces and in a few biographies, precise in statement, artistic in diction and style, broadly comprehensive, yet vividly suggestive. The others attain nearly the same degree of interest, Professor Greenhill's by its graphic mingling of experiment with theory, and Professor Wirtinger's by its lucid exposition of rich material preserved in a shorthand report of Riemann's lectures, and now brought to light after more than forty years of oblivion. After reading this and recalling the few similar events now classic, one may venture to expect that some day yet the missing manuscripts of Jacobi, alluded to by Königsberger, may be recovered.

The titles of the papers read before the six sections, nearly eighty in number, were given in the BULLETIN a year ago,* together with many abstracts and summaries. Most of them will appeal to large circles of readers; it may be permissible to mention one in particular that is universally interesting, that of Professor F. Meyer, of Königsberg, on the essential nature of mathematical proof. That there is no real increase of mathematical knowledge has been affirmed by others, but certainly not often reinforced by such a convincing series of far-reaching analyses: to the principal thesis the eloquent closing paragraph may be found to supply the needful antithesis.

H. S. WHITE.

NOTES.

THE April number (volume 7, number 2) of the *Transactions of the* AMERICAN MATHEMATICAL SOCIETY contains the following papers: "On geometries in which circles are the shortest lines," by C. E. STROMQUIST; "A generalization of the notion of angle," by G. A. BLISS; "The square root and relations of order," by O. VEBLEN; "The problem of partial geodesic representation," by E. KASNER; "On the pentadeltoid," by R. P. STEPHENS; "The groups of order p^m which contain exactly p cyclic subgroups of order p^a" by G. A. MILLER; "Groups in which a large number of operators may correspond to their inverses," by W. A. Manning; "Finite projective geometries," by O. Veblen and W. H. Bussey; "On the analytic extension of functions defined by double power series," by W. B. Ford; "On quadratic hermitian and bilinear forms," by L. E. Dickson; "Die kinematische Erzeugung von Minimalflächen," by P. Stäckel; "A fifth necessary condition for a strong extremum of the integral

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* Vol. 11 (1905), pp. 205–217, and 247–263.

$$\int_{x_0}^{x_1} F(x, y, y')dx,"$$

by O. Bolza; "A problem in the calculus of variations in which the integrand is continuous," by G. A. Bliss and M. Mason.

The April number (volume 7, number 3) of the *Annals of Mathematics* contains: "Introduction to the theory of Fourier's series (concluded)," by M. Bôcher; "Note on multiply perfect numbers," by R. D. Carmichael; "Note on integrating factors," by E. B. Wilson.

At the meeting of the London mathematical society held on March 8, the following papers were read: By H. Lamb, "Sommerfeld's diffraction problem and reflection by a parabolic mirror"; by L. J. Rogers, "Function-sum theorems connected with the series

$$\sum x^n/n^2";$$

by T. J. I'A. Bromwich, "Investigations on series of zonal harmonics"; by E. W. Barnes, "The functions $g_\beta(x, \theta)$ and $f_\beta(x, \theta)$"; by E. J. Nanson, "Relations between the p-line determinants formable from a p by q array."

The active participation of the members of the American Mathematical Society in the organization of associations of teachers of mathematics is partially indicated by the following list of members who are officers of these associations in the middle west or on the Pacific coast: Professor H. C. Harvey, president, Dr. L. D. Ames, secretary, and Professor E. R. Hedrick, chairman of the executive council in the Missouri association; Professor T. E. McKinney, secretary of the Ohio association; Professor W. J. Rusk, chairman of the mathematics section of the Iowa teachers' association; Professor H. E. Slaught, vice-president of the mathematics section of the Central association of science and mathematics teachers; Professor G. A. Miller, president of the mathematics section of the California teachers' association; Professor R. E. Moritz, president of the Washington state association of mathematics teachers.

THE seventy-sixth annual meeting of the British association for the advancement of science will be held at York, August 3 to 8. Professor R. LANKESTER is president of the association, and Mr. E. H. GRIFFITHS is president of section A, mathematics and physics.

THE publishing house of Gauthier-Villars in Paris announces the following books in press: J. TANNERY, Leçons d'algebre et d'analyse, volume II; P. JANET, Leçons d'electrotechnique générale, volume III; E. PICARD, Oeuvres de Charles Hermite, volumes II and III.

UNIVERSITY OF CHICAGO.— The following advanced courses in mathematics are announced for the summer quarter, June 19 – September 1. — By Professor O. BOLZA: Elliptic functions, four hours; Functions of a real variable, four hours. — By Professor H. MASCHKE: Projective geometry, four hours. — By Professor H. E SLAUGHT: Elliptic integrals, four hours; Analytic geometry, five hours. — By Professor L. E. DICKSON: Algebraic analysis, four hours; Theory of substitutions, four hours. — By Dr. A. C. LUNN: Integral calculus, five hours; General Seminar, two hours. — By Mr. N. J. LENNES: Pedagogy of mathematics, four hours.

YALE UNIVERSITY. — The following advanced courses are announced for the year 1906-1907. — By Professor J. PIERPONT: Advanced mechanics, two hours; Advanced theory of functions, two hours; Theory of functions of a real variable, two hours. — By Professor P. F. SMITH: Advanced analytic geometry, two hours; Foundations of geometry, one hour. — By Professor H. E. HAWKES: Linear associative algebra, two hours; Teachers course in geometry, two hours. — By Professor M. MASON: Calculus of variations, two hours; Differential equations of physics, two hours. — By Professor E. B. WILSON: Advanced calculus, two hours; Thermodynamics, two hours. — By Dr. W. A. GRANVILLE: Differential geometry, two hours. — By Dr. L. E. HEWES: Differential equations, one hour; Geometric transformations, two hours. — By Mr. E. L. TAYLOR: Scientific computation, one hour. — By Professor W. B. BEEBE: Celestial mechanics, two hours. — By Professor F. E. BEACH: Vector analysis, one hour (first half year).

THE foreign universities below offer courses in advanced mathematics during the present summer semester as follows:

OXFORD UNIVERSITY (Easter and Trinity terms). — By Professor W. ESSON: Comparison of analytic and synthetic methods in the theory of conics, two hours; Informal instruction in geometry, one hour. — By Professor E. B. ELLIOTT: Theory of functions, I, three hours. — By Professor A. E. H. LOVE: Waves and sound, two hours. — By Mr. A. L. DIXON: Calculus of variations, one hour. — By Mr. H. T. GERRANS: Line geometry, two hours. — By Mr. A. E. JOLLIFFE: Higher analytic geometry, two hours. — By Mr. P. J. KIRKBY: Plane curves, two hours. — By Mr. J. W. RUSSELL: Rigid dynamics, two hours. — By Mr. R. F. MCNEILE: Algebra, two hours. — By Mr. C. E. HASELFOOT: Series and continued fractions, two hours. — By Mr. A. L. PEDDER: Spherical trigonometry, one hour. — By Mr. C. H. SAMPSON; Solid geometry, two hours. — By Mr. C. H. THOMPSON: Differential equations, two hours.

UNIVERSITY OF PARIS. — By Professor É. PICARD: Integrals of differential equations, two hours. — By Professor E. GOURSAT: Ordinary and partial differential equations, two hours. — By Professor L. RAFFY: Applications of partial differential equations to geometry, two hours. — By Professor P. PAINLEVÉ: Analytic mechanics, two hours. — By Professor P. APPELL: Rational mechanics, two hours. — By Professor É. BOREL: Theory of integral functions, one hour. Each professor will hold a weekly conference in mathematics, assisted by Professor J. HADAMARD, Dr. E. BLUTEL and Dr. SERVANT.

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Analytic functions, four hours; Surfaces and twisted curves, four hours; Exercises in maxima and minima, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of determinants, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Differential calculus, four hours; Abelian and theta functions, two hours; Seminar, three hours. — By Professor J. KNOBLAUCH: Applications of elliptic functions, four hours; Analytic geometry, four hours; Line congruences, one hour. — By Professor G. HETTNER: Theory of probabilities, two hours. — By Professor R. LEHMAN-FILHÉS: Analytic mechanics, four hours. — By Professor E. LANDAU: Picard's theorem, four hours. — By Dr. I. SCHUR: Integral calculus, four hours; with exercises, one hour.

UNIVERSITY OF BRESLAU. — By Professor J. ROSANES: Plane analytic geometry, three hours; Seminar, one hour. — By Professor R. STURM: Differential geometry, four hours; Descriptive geometry, two hours; Seminar, one hour. — By Professor A. KNESER: Integral calculus, four hours; with exercises, two hours; Calculus of variations, three hours; Seminar. two hours. — By Professor G. LANDSBERG: Differential calculus, four hours; with exercises, two hours; Elliptic and modular functions, three hours.

UNIVERSITY OF FREIBURG. — By Professor J. LÜROTH: Integral calculus, five hours; Trigonometry, two hours; Seminar, two hours. — By Professor L. STICKELBERGER: Analytic mechanics, five hours; Fourier's series and integrals, two hours; Seminar, two hours. — By Professor A. LOEWY: Theory and application of determinants, four hours; Foundations of geometry, two hours; Exercises in the mathematics of insurance. — By Professor J. WEINGARTEN: Selected chapters from the theory of elastic bodies, two hours. — By Dr. K. SEITH: Descriptive geometry, two hours; with exercises.

UNIVERSITY OF GIESSEN. — By Professor M. PASCH: Algebra, four hours; Invariants, three hours; Seminar, one hour. — By Professor E. NETTO: Plane analytic geometry, four hours; Definite integrals, three hours; Seminar, one hour. — By Professor H. GRASSMANN: Analytic mechanics, I, four hours.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Theory of functions, four hours; Seminar (with Professors Hilbert and Minkowski), two hours. — By Professor D. HILBERT: Differential and integral calculus, four hours; Mechanics of continua, four hours. — By Professor H. MINKOWSKI: Algebra, four hours; Spherical harmonics, two hours. — By Professor C. RUNGE: Differential equations, six hours; Seminar, two hours. — By L. PRANDTL: Graphical methods, two hours. — By Professor M. BRENDEL: Theory of probabilities, four hours; Mathematics of insurance, two hours; Seminar, two hours. — By Professor E. ZERMELO: Partial differential equations of mathematical physics, four hours. — By Dr. M. ABRAHAM: Theory of potential, four hours. — By Dr. G. HERGLOTZ: Analytic geometry, four hours. — By Dr. C. CARATHEODORY: Calculus of variations, four hours.

UNIVERSITY OF GREIFSWALD. — By Professor W. THOMÉ:

Theory of analytic functions, elliptic functions, four hours; Differential geometry, two hours; Seminar, two hours. — By Professor F. ENGEL : Analytic mechanics, I, four hours; Analytic geometry of two and three dimensions, four hours; Differential invariants, one hour; Seminar, two hours.—By Professor K. TH. VAHLEN : Integral calculus, four hours; with exercises, one hour; Determinants, one hour.

UNIVERSITY OF KIEL. — By Professor L. POCHHAMMER : Theory of determinants, four hours; Application of the calculus, four hours; Seminar, one hour. — By Professor L. HEFFTER : Differential and integral calculus, four hours; with exercises, one hour; Algebra, four hours; Applications of elliptic functions, one hour.

UNIVERSITY OF KÖNIGSBERG.—By Professor W. F. MEYER : Plane analytic geometry, three hours; Introduction to higher geometry, four hours; Seminar, two hours. — By Professor A. SCHOENFLIES : Theory of functions, five hours; Seminar, two hours. — By Professor L. SAALSCHÜTZ : Determinants, two hours; Differential calculus, four hours; with exercises, two hours.

UNIVERSITY OF MARBURG. — By Professor K. HENSEL : Theory of numbers, four hours; Differential equations, four hours; Seminar, one hour. — By Professor E. NEUMANN : Elliptic functions, four hours; Fourier's series, two hours; Seminar, two hours. — By Dr. F. v. DALWIGK : Analytic geometry, four hours. — By Dr. H. JUNG : Algebraic solution of equations, four hours.

UNIVERSITY OF MÜNSTER. — By Professor W. KILLING : Analytic geometry, I, four hours; with exercises, two hours; Seminar, one hour. — By Professor R. v. LILIENTHAL : Differential and integral calculus, four hours; Curvature of curves and surfaces, four hours. —By Professor M. DEHN : Mechanics, four hours; Analysis situs, one hour; Seminar, two hours.

UNIVERSITY OF STRASSBURG. — By Professor TH. REYE : Synthetic geometry, two hours; Mechanics, four hours; Seminar, two hours. — By Professor H. WEBER : Calculus and introduction to the theory of functions, four hours; Advanced theory of numbers, three hours; Seminar, two hours. — By Professor M. SIMON : History of mathematics in the middle ages, three hours. — By Professor J. WELLSTEIN : Introduction to the theory of invariants, two hours; Encyclopedia of elementary mathematics, II, three hours; Seminar, two hours. — By Professor E. TIMERDING: Analytic geometry of space, three hours; Applied mechanics, three hours; Theory of probabilities, one hour. — By Professor S. EPSTEIN: Elliptic functions, two hours; Seminar, two hours.

UNIVERSITY OF TÜBINGEN. — By Professor A. v. BRILL: Mechanics, five hours; Foundations of geometry, three hours; Seminar, two hours. — By Professor H. v. STAHL: Lower analysis, three hours; Theory of functions, three hours; Seminar, two hours. — By Professor L. MAURER: Higher analysis, I, three hours; with exercises, two hours; Linear differential equations, one hour.

UNIVERSITY OF BASEL. — By Professor H. KINKELIN: Applications of the calculus, three hours; Analytic geometry, three hours; Mathematics of insurance, one hour. — By Professor O. SPIESS: Special differential equations, three hours; Determinants, one hour. — By Professor M. GROSSMANN: Descriptive geometry, two hours; Non-euclidean geometry, two hours.

UNIVERSITY OF GENEVA. — By Professor C. CAILLER: Differential and integral calculus, three hours; Rational mechanics, three hours; Seminar, two hours. — By Professor H. FEHR: Projective and descriptive geometry, two hours; Higher algebra, two hours; Seminar, one hour. — By Dr. R. DE SAUSSURE: Geometry of motion, two hours.

PROFESSOR D. HILBERT has been awarded the Cothenius medal of the academy of sciences at Halle.

PROFESSOR E. C. PICKERING, director of the observatory at Harvard University, has been elected a member of the Berlin academy of sciences.

PROFESSOR G. HUBER has been promoted to a full professorship of mathematics at the University of Bern, Switzerland.

PROFESSOR J. H. HAGEN, director of the observatory of the University of Georgetown, has been appointed director of the observatory of the Vatican.

DR. W. SCHLINK, of the technical school at Darmstadt, has been appointed professor of mechanics at the technical school at Braunschweig.

UPON the invitation of King Oscar, Professor V. VOLTERRA, of the University of Rome, gave a series of lectures on the differential equations of mathematical physics at the University of Stockholm during the month of February.

DR. VANECEK has been appointed docent in mathematics at the Bohemian technical school at Prague.

DR. G. Z. GIAMBELLI has been appointed docent in projective geometry at the University of Genoa.

DR. G. DE FRANCHIS has been appointed to an associate professorship of algebra and analytic geometry at the University of Cagliari.

PROFESSOR G. FUBINI, of the University of Catania, has been transferred to the University of Genoa, as associate professor of the calculus.

PROFESSOR OSKAR BOLZA, of the University of Chicago, who is spending the year in travel and study in southern Europe and in Egypt, will resume his lectures at the university at the beginning of the summer quarter, June 19, 1906.

MR. L. A. MARTIN, JR., has been promoted to an assistant professorship of mathematics and mechanics at the Stevens Institute of Technology.

PROFESSOR E. L. RICHARDS, of Yale University, will retire from active service at the close of the present academic year.

PROFESSOR J. M. PEIRCE, Perkins professor of mathematics and astronomy and senior member of the faculty of Harvard University, died March 21, at the age of 71 years. He was appointed tutor in mathematics in 1854, university professor of mathematics in 1869, and Perkins professor of mathematics and astronomy in 1885. Professor Peirce had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1898.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Vita matematica napoletana. Studio storico, biografico, bibliografico. Parte prima. Napoli, Giannini, 1905. 8vo. 216 pp. L. 10.00

BACHET (C. G.). Problèmes plaisants et délectables qui se font par les nombres. 4e édition, revue et simplifiée. Paris, Gauthier-Villars, 1905. 16mo. 6 + 161 pp. Fr. 3.50

BAUER (G.). Von der Kurve sechster Ordnung, welche der Ort der Brennpunkte der Kegelschnitte ist, welche durch vier Punkte gehen. München, 1905. 8vo. 11 pp.

BERRY (W. J.). Differential equations of the first species. New York, Van Nostrand, 1906. 12mo. Cloth.

BIANCHI (L.). Teoria delle trasformazioni delle superficie applicabili sulle quadriche rotonde. Roma, 1905. 4to. 74 pp.

BÜCHEL (C.). Ganzzahlige Werte bei Diophant. Den Teilnehmern der 48sten Versammlung der Philologen und Schulmänner zu Hamburg 1905 als Festgabe dargeboten von dem Ortskomitee. Hamburg, Herold, 1905. 8vo. 16 pp. M. 1.00

COUTURAT (L.). Les principes des mathématiques, avec un appendice sur la "Philosophie des mathématiques" de Kant. Paris, Alcan, 1905. 8vo. 8 + 311 pp. Fr. 5.00

CZUBER (E.). Vorlesungen über Differential- und Integralrechnung. 1er Band, 1te Hälfte. 2te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1906. 8vo. 256 pp. M. 6.00

FASSBINDER (C.). Théorie et pratique des approximations numériques. Paris, Gauthier-Villars, 1906. 8vo. 6 + 92 pp. Fr. 3.00

GOMES TEIXEIRA (F.). Tratado de las curvas especiales notables. Memoria premiada por la Real Academia de Ciencias Exactas, Fisicas y Naturales de Madrid, y publicada por la misma Academia. Madrid, 1905. 9 + 632 pp.

HALDANE (E. S.). Descartes, his life and times. New York, Dutton, 1905. 8vo. 28 + 398 pp. Cloth. $4.50

HAMBURGER (A.). Ueber die Restabschätzung bei asymptotischen Darstellungen der Integrale linearer Differentialgleichungen zweiter Ordnung. (Diss.) Berlin, 1905. 8vo. 58 pp.

HILL (G. W.). Collected mathematical works. (In four volumes.) Vol. 2. Washington, Carnegie Institution, 1904. 4to. 5 + 339 pp. Cloth. $2.50

JACOBSTHAL (E.). Anwendungen einer Formel aus der Theorie der quadratischen Reste. (Diss.) Berlin, 1906. 8vo. 39 pp.

MACAULAY (F. S.). Geometrical conics. 2d edition. Cambridge, University Press, 1906. 8vo. 310 pp. Cloth. 4s. 6d.

MUIR (J.). Theory of determinants; in the historical order of development. Part 1: General determinants up to 1841. Part 2: Special determinants up to 1841. 2d edition, revised. London, Macmillan, 1906. 8vo. 504 pp. Cloth. 17s.

OSGOOD (W. F.). Lehrbuch der Funktionentheorie. (In 2 Bänden.) 1er Band, 1te Hälfte. Leipzig, Teubner, 1906. 8vo. 306 pp. M. 7.00

SAUER (R.). Eine polynomische Verallgemeinerung des Fermatschen Satzes. (Diss.) Giessen, 1905. 8vo. 18 pp.

Simon (M.). Methodik der elementaren Arithmetik in Verbindung mit algebraischer Analysis. Leipzig, Teubner, 1906. 8vo. 4 + 108 pp.

Tesař (L.). Elemente der Differential- und Integralrechnung. Hilfsbuch für mathematischen Unterricht zum Gebrauche an höheren Lehranstalten. Leipzig, Teubner, 1906. 8vo. 8 + 128 pp.

Thalreiter (F.). Auflösung gewisser algebraischer Eliminationsaufgaben durch Benützung der Teilungsgleichungen der p-Funktion. (Diss.) München, 1905. 8vo. 59 pp.

Thomae (J.). Grundriss einer analytischen Geometrie der Ebene. Leipzig, Teubner, 1906. 8vo. 10 + 184 pp.

Toeplitz (O.). Ueber Systeme von Formen, deren Funktionaldeterminante identisch verschwindet. (Diss.) Breslau, 1905. 8vo. 46 pp.

Valewink (G. C. A.). Over asymptotische ontwikkelingen. Utrecht, 1905. 8vo. 138 pp. M. 3.00

Vreeswijk (J. A.). Involuties op rationale krommen. Utrecht, 1905. 8vo. 104 pp. M. 2.50

Vogt (H.). Eléments de mathématiques supérieures, à l'usage des physiciens, chimistes et ingénieurs, et des élèves des facultés des sciences. 3e édition. Paris, Vuibert, 1906. 8vo. 7 + 620 pp. Fr. 10.00

Wallner (C. R.). Die Verteilung der Primzahlen nach neuen Gesichtspunkten behandelt. (Diss.) München, 1905. 8vo. 53 pp.

Westfall (W. D. A.). Zur Theorie der Integralgleichungen. (Diss.) Göttingen, 1905. 8vo. 67 pp.

Winkler (F.). Die infinitesimalen Transformationen, welche einen Pfaffschen Ausdruck absolut oder modulo eines vollständigen Differentials invariant lassen. (Diss.) Leipzig, 1905. 8vo. 38 pp. M. 1.50

Wünschmann (K.). Ueber Berührungsbedingungen bei Integralkurven von Differentialgleichungen. (Diss., Greifswald.) Leipzig, Teubner, 1905. 8vo. 36 pp.

II. ELEMENTARY MATHEMATICS.

Baumberger (G.). Lehrbuch der Planimetrie mit Aufgabensammlung für Mittelschulen, insbesondere technische Lehranstalten. Bern, Stämpfli, 1906. 8vo. 6 + 82 pp. M. 1.60

Bozal Ovejero (A.). Tratado elemental de geometría. Madrid, Hernando, 1905. 8vo. 296 pp. Fr. 10.00

Cob (A.). See Vásquez Figueroa (A.).

Dalman y Carles (J.). Aritmética razonada y nociones de algebra. Tratado teórico-práctico demostrado con aplicación á las diferentes cuestiones mercantiles para uso de las escuelas normales y de las de comercio. Comprende, además de la teoría indispensable, más de 5000 ejercicios y problemas aritméticos, algebraicos y geométricos para el cálculo mental y escrito. 11a edición corregida. Libro del alumno. Grado profesional. Gerona, Dalman, Carles, y Ca, [1906]. 8vo. 515 pp. Fr. 6.50

Deakin (R.). Algebra, preliminary certificate edition, with a supplement on graphs. London, Clive, 1906. 8vo. 482 pp. Cloth. 3s.

Elsässer (W.). Leitfaden der Stereometrie. Ein Hilfsbuch zum Gebrauch beim Unterricht an höheren Lehranstalten. Stuttgart, Grub, 1906. 8vo. 8 + 90 pp. M. 1.50

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THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and twenty-eighth regular meeting of the Society was held in New York City on Saturday, April 28. The following fifty members attended the sessions:

Dr. Grace Andrews, Professor G. A. Bliss, Professor Maxime Bôcher, Professor C. L. Bouton, Professor E. W. Brown, Dr. W. H. Bussey, Professor F. N. Cole, Miss E. B. Cowley, Miss L. D. Cummings, Dr. W. S. Dennett, Professor L. P. Eisenhart, Professor B. F. Finkel, Professor T. S. Fiske, Professor G. H. Hallett, Professor E. R. Hedrick, Dr. L. I. Hewes, Mr. A. M. Hiltebeitel, Professor E. V. Huntington, Mr. S. A. Joffe, Dr. Edward Kasner, Professor C. J. Keyser, Dr. G. H. Ling, Mr. L. L. Locke, Professor E. O. Lovett, Dr. Emory McClintock, Professor James Maclay, Professor H. P. Manning, Professor Max Mason, Mr. A. R. Maxson, Professor W. F. Osgood, Professor James Pierpont, Mr. R. G. D. Richardson, Mr. W. H. Roever, Miss Ida M. Schottenfels, Mr. C. H. Sisam, Dr. Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Dr. H. F. Stecker, Dr. C. E. Stromquist, Professor H. D. Thompson, Mr. M. O. Tripp, Professor H. W. Tyler, Professor Oswald Veblen, Professor L. A. Wait, Mr. H. E. Webb, Professor J. B. Webb, Professor H. S. White, Dr. Ruth G. Wood.

President W. F. Osgood occupied the chair. The Council announced the election of the following persons to membership in the Society: Rev. R. D. Carmichael, Hartselle, Ala.; Mr. F. L. Griffin, University of Chicago; Mr. W. R. Longley, University of Chicago; Mr. W. D. MacMillan, University of Chicago; Mr. F. W. Owens, Evanston Academy; Dr. J. J. Quinn, High School, Warren, Pa.; Mr. W. J. Risley, University of Illinois; Dr. R. P. Stephens, Wesleyan University; Mr. J. D. Suter, Iowa State College; Mr. A. M. Wilson, McKinley High School, St. Louis, Mo. Eighteen applications for membership in the Society were received.

Professor W. F. Osgood was elected a member of the Editorial Committee of the *Transactions*, to succeed Professor E. W. Brown, who retires after seven years' service covering the entire period of existence of that journal.

The by-laws were amended to provide that only members of at least four years' standing shall be permitted to compound life membership.

In the interval between the sessions the members lunched together, and the informal dinner in the evening, attended by some thirty members, afforded another welcome opportunity for conference and renewal of acquaintance.

The following papers were read at this meeting:

(1) Professor G. A. MILLER: "Groups in which all the operators are contained in a series of subgroups such that any two of them have only identity in common."

(2) Mr. W. H. ROEVER: "Lines of force illustrated by rotating carriage wheels."

(3) Mr. W. H. ROEVER: "Systems of lines of force whose differential equations take Bernoulli's form in polar coordinates."

(4) Professor VIRGIL SNYDER: "On twisted curves contained in a linear complex."

(5) Mr. G. E. WAHLIN: "On the number of classes of binary quadratic forms and the ideals of a quadratic body."

(6) Mr. R. G. D. RICHARDSON: "On the fundamental theorem in the reduction of multiple integrals."

(7) Professor JAMES PIERPONT: "On the area of curved surfaces."

(8) Professor E. R. HEDRICK: "Functions and their derivatives on given assemblages."

(9) Professor E. R. HEDRICK: "Lipschitz's condition in the case of implicit functions."

(10) Professor MAX MASON: "A necessary condition for an extremum of a double integral."

(11) Professor G. A. BLISS: "An invariant of the calculus of variations corresponding to geodesic curvature."

(12) Dr. EDWARD KASNER: "A generalization of conformal representation."

(13) Dr. EDWARD KASNER: "Velocity curves in the dynamics of a particle."

(14) Professor J. W. YOUNG: "A generalization of a problem of Tchebychev."

(15) Professor C. J. KEYSER: "Concerning the bond uniting elements into a space."

(16) Dr. C. N. HASKINS: "Note on the differential invariants of a plane."

(17) Mr. E. C. Colpitts: "On twisted quintic curves."

(18) Mr. W. C. Breuke: "On the differentiation of trigonometric series" (preliminary communication).

(19) Dr. I. E. Rabinovitch: "The necessary and sufficient kinematic axioms of geometry."

Mr. Wahlin was introduced by Professor Pierpont, Dr. Rabinovitch by Professor Cole; Mr. Colpitts's paper was communicated through Professer Snyder. In the absence of the authors, the papers of Professor Miller and Professor Keyser were read by title, those of Dr. Haskins and Mr. Colpitts were presented by Professor Snyder, and Mr. Breuke's paper by Professor Bôcher. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Miller's paper appears in full in the present number of the Bulletin.

2. Mr. Roever illustrated, by means of the crossings of spokes of two wagon wheels rotating in parallel planes, the lines of force of a system composed of two sources (of the same or different intensities and of the same or different algebraic signs). The hubs of the wheels correspond to the sources, and the ratio of the angular velocities of the wheels is the negative reciprocal of the ratio of the intensities of the sources. In particular, he considered the case in which a rolling wheel is viewed through a picket fence, and explained why it is that the wheel appears to turn backwards.

3. In this paper Mr. Roever considered the field of force which results from the combination of the fields I and II described below.

Field I. At any point P the direction of the force due to this field passes through a fixed point O, and the magnitude of the force is represented by f_1.

Field II. At any point P the direction of the force due to this field is parallel to the fixed direction OY, and the magnitude is represented by f_2.

If we denote by r the distance OP, and by θ the complement of the angle YOP, the differential equation of the lines of force of the resulting field is

$$(1) \qquad \frac{dr}{d\theta} - \tan\theta \cdot r = -\frac{f_1}{f_2} r \sec\theta.$$

If

$$\frac{f_1}{f_2} = r^n F(\theta),$$

equation (1) takes Bernoulli's form $(dy/dx + P(x)\cdot y = Q(x)\cdot y^n)$.

As applications, Mr. Roever solved the following problems.

(1) Let it be required to find the lines of force of the system which is composed of a uniform electric field of intensity 2κ and an electrified point of charge M. Here

$$f_1 = \frac{m}{r^2}, \quad f_2 = 2\kappa, \quad \frac{f_1}{f_2} = \frac{m}{2\kappa}\frac{1}{r^2}.$$

(2) Let it be required to find the lines of force of the system which is composed of a uniform electric field of intensity 2κ, and an electrified straight line (of infinite length) of charge λ per unit length, the straight line being perpendicular to the lines of force of the uniform field. Here

$$f_1 = \frac{2\lambda}{r}, \quad f_2 = 2\kappa, \quad \frac{f_1}{f_2} = \frac{\lambda}{\kappa}\frac{1}{r}.$$

(3) Assuming the earth to be a rigid sphere with its center of mass at its center of figure, let it be required to find the lines of force of the system which is composed of the gravitational field and the field due to the centrifugal force of rotation. This is the field of force which determines the direction of the plumb line. Here

$$f_1 = \frac{M}{r^2}, \quad f_2 = \omega^2 y = \omega^2 r \sin\theta, \quad \frac{f_1}{f_2} = \frac{M}{\omega^2}\frac{1}{r^3 \sin\theta},$$

where ω is the angular velocity of rotation and M is the mass of the earth.

4. Curves belonging to a linear complex can be depicted into developables containing a conic by means of the Noether point-line transformation. The latter are transformed by duality into curves lying on a quadric cone. In Professor Snyder's paper the details of these transformations are worked out, and a classification of curves contained in a linear complex is given. The simplest such curve of genus one is of order six, of genus two of order seven, of genus five of order eight, etc.

5. Mr. Wahlin presented a simultaneous development of the two theories in question, showing how, by applying the ideal theory to the forms, we are able to simplify the evaluation of the number of classes. He also gave an application of the ideals to the determination of the number of genera of forms.

6. In the *Journal de Mathématiques*, series 4, volume 8 (1892), de la Vallée-Poussin proposes the following problems: Is the existence of the double integral over a plane field T,

$$\int_T f(x, y)dxdy,$$

and of the iterated integral

$$\int_Y dy \int_X f(x, y)dx$$

a sufficient condition for the equality of these integrals? He proves that such is the case if certain restrictions are put on the arrangement of the infinities of the function $f(x, y)$. In the same journal, series 5, volume 5 (1899), he devotes a memoir to this problem, and shows that under certain limitations of the function the theorem is true. He states, however, that in his opinion the theorem is true without these restrictions. Mr. Richardson proved that the existence of these integrals is a sufficient condition for their equality and that the condition of uniform or regular convergence of the integral

$$\int_X f(x, y)dx$$

is entirely unnecessary. It is shown that this theorem is also true for the generalized integrals introduced by Professor Pierpont. Further, if the fields of integration are $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, of $m + n$, m, and n dimensions respectively, then the existence of

$$\int_{\mathfrak{A}} f(x_1, x_2, \cdots, x_{m+n}) \quad \text{and} \quad \int_{\mathfrak{B}}\int_{\mathfrak{C}} f(x_1, x_2, \cdots, x_{m+n})$$

is a sufficient condition for their equality.

7. Professor Pierpont extended the results of Stolz (*Transactions*, volume 3 (1902), page 23) in a two-fold manner.

First, he considered surfaces whose coordinates do not have first partial derivatives, finite or infinite, at a discrete aggregate of points in the parametric field. Secondly, he considered a much wider system of inscribed polyhedra than is employed by Stolz.

8. In this paper Professor Hedrick discusses the properties of functions of one or more real variables which are defined for the values of the independent variable or variables which constitute a given assemblage. The continuity of such functions, the possibility of adjoining further definition, definitions and theorems concerning continuity, approach to limits, etc., form the first part of the paper.

The discussion of derivatives of such functions leads to an important notion of the derivative of any function on an assemblage, and in particular to the notion of sequence derivatives. After a discussion of these concepts, a general theorem is proved, namely, that if a continuous sequence derivative exists, the ordinary derivative exists and is continuous.

An extension of these results to a generalized notion of derivative and to the concept of a jacobian as an assemblage derivative in space closes the paper.

9. In this paper Professor Hedrick develops the analogon to Lipschitz's condition for the existence of solutions of systems of differential equations, and shows the form which the condition assumes in the case of the solutions of implicit equations. The form of proof given recently by Goursat is used as a basis, but the conditions are materially weakened. In particular the Goursat condition which requires the existence of $\partial F(x, y)/\partial y$ (for an implicit equation $F(x, y) = 0$ which is to be solved for y) is replaced by the weaker condition that the difference quotient

$$Q(x, \bar{y}, \bar{\bar{y}}) \equiv \frac{F(x, \bar{y}) - F(x, \bar{\bar{y}})}{\bar{y} - \bar{\bar{y}}}$$

should lie between two positive numbers (or two negative numbers) for all points near the known point (x_0, y_0) which is on $F(x, y) = 0$. The generalization to n equations is given in several forms.

10. If a function $z(x, y)$ gives an extreme value (maximum or minimum) to the integral

$$J = \int\int f(x, y, z, p, q)\, dxdy \qquad (p = z_x,\ q = z_y),$$

then the first variation of J must vanish, and the second variation must have the same sign for all allowed variations of z. Professor Mason showed that the second variation could be made positive or negative at pleasure, unless

$$f_{pp} f_{qq} - f_{pq}^2 \geqq 0 .$$

This inequality is accordingly a necessary condition for an extremum.

11. The geodesic curvature of a curve on a given surface is usually defined by means of the projection of the given curve on a tangent plane to the surface, but it may also be defined in other ways. For example, if two geodesic lines are drawn tangent to the curve at neighboring points A and A', they will in general general intersect at a point B. The limit as A' approaches A of the ratio of the angle at B between the two geodesics, divided by the length of the arc AA', is the geodesic curvature at the point A.

In the paper of Professor Bliss a generalization of geodesic curvature is found by making use of the definition just given. A problem of the calculus of variations is considered in which the integral has the form

$$I = \int f(x, y, \tau) \sqrt{x'^2 + y'^2}\, dt,$$

where x and y are functions of t defining the given curve, and τ is the angle between the tangent to the curve and the x-axis. In a previous paper the writer has given a generalization corresponding to this integral. If two extremals are taken tangent to the given curve at neighboring points A and A', they will in general intersect, and the generalized angle between them may be determined. The limit of the ratio of this angle to the value of I taken along the arc AA', as A' approaches A, is the desired curvature. It turns out to be an invariant under point transformation.

12. In a previous paper, Dr. Kasner has shown that if a surface is mapped conformally on a plane, the geodesics through any point are pictured by curves whose circles of curvature at the common point form a pencil, the locus of centers of curvature thus being a straight line. In the present paper it is shown that this property belongs to a very general class of representations which includes the conformal class as a special case. The representation is next required to have an additional property, also possessed by conformal representations: the curves on the surface which are depicted by the straight lines $x =$ const., $y =$ const., are to form an orthogonal net. It is shown that non-conformal representations having both properties arise when, and only when, the surface considered is of the Liouville type. The paper concludes with a discussion of possible representations of a plane on itself.

13. Dr. Kasner's second paper relates to the motion of a particle in a plane under the action of any force depending only upon the particle's position. As the particle describes a trajectory its velocity in general varies. For each set of values of x, y, y' v, a unique trajectory is defined, and therefore, at the point in question, a unique value of the curvature or, what is equivalent, of y''. It is thus possible to express the speed v in terms of x, y, y', y''. If now v is replaced by a constant, we have a differential equation of the second order. The curves satisfying this equation are termed "velocity curves." To each speed there corresponds a doubly infinite system of these curves. The first part of the paper investigates these systems, both in general and for conservative, solenoidal, and Laplacian forces. Relations arise to the theory of geodesics, and to isogonal trajectories.

The second part of the paper deals with the total system of ∞^3 velocity curves, corresponding to all values of v. Such a system can never be the system of trajectories either of the same force or of another force. Many of the geometric properties obtained are analogous to the properties of systems of trajectories obtained in a previous paper (see *Transactions* for July, 1906). The properties relate to osculating parabolas and hyperosculating circles; the whole set is completely characteristic.

14. Tchebychev considered the problem of determining, if it exists, the polynomial of given degree n which approximates

most closely to given continuous function of the real variable x in a given finite interval. The generalization considered in Professor Young's paper consists in replacing the given function by an arbitrary function $f(x)$ which is required merely to satisfy a certain functional relation $\phi[f(x)] = 0$ of a general character. The existence of at least one polynomial of (closest) approximation of degree n with reference to ϕ is proved under very general hypotheses on the function ϕ. Under somewhat less general hypotheses, a certain necessary condition is derived that a polynomial of degree n be a polynomial of approximation; and this condition is shown also to be sufficient provided a certain determinant does not vanish. With this same restriction, it is shown that the solution is unique, *i. e.*, that there exists only one polynomial of approximation of degree n with reference to a given ϕ.

15. Professor Keyser's paper considers the question: What, if any, analytic fact corresponds to the universal conviction of "natural" man that the points of (our) space constitute one, a whole—(our) space? The hypothesis is suggested as such that any continuous space ${}_nS_e$ of dimensionality n and of elements e of a kind is constituted a space by virtue of some relation r (different for different S's) subsisting between each e and an entity E (different for different S's) itself not an element of (not *in*) ${}_nS_e$. The hypothesis, naturally not admitting of mathematical demonstration, is nevertheless indicated independently by an infinitude of facts and is not inconsistent with any fact of which the writer is aware. Of such indications the following one may be cited as being at once simple and typical. Denote by S_s the ensemble of spheres orthogonal to a sphere S. The elements of S_s are spheres. Let M be an inhabitant of S_s and be conformed to S_s as the intuition of a human intelligence H is conformed to S_p, ordinary point space. Suppose M and H to write each his own geometry and by chance to employ throughout the same nomenclature. Suppose their works exchanged. Then each may read his own geometry in the other's book. M is certain that his elements (spheres) constitute space. H has precisely the same conviction (feeling) with respect to his elements (points). Neither M nor H perceives the ground of his own conviction. H, however, perceives the (at least *a*) reason for M's conviction. It is that each of M's elements bears a certain same relation r (orthogonality) to a

certain entity E (sphere S) not in M's space S_4. Reciprocity demands a corresponding perception by M in the case of S_p. Like analogies abound. To an inhabitant of a space, to an intelligence of intuition conformed to it, the space as an entity could not appear, as for example a plane dweller could not behold the plane. In case of S_p, E is probably S_p itself as the latter would appear to a beholder outside; and r probably is distantial; specifically, the distance zero of each point from E, and so of the form

$$(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5) : \sqrt{a_1^2 + \cdots + a_4^2} = 0.$$

16. The paper of Dr. Haskins calls attention to an error in the recent paper of Professor Forsyth on "Differential invariants of a plane and of curves in the plane." Professor Forsyth states: (1) that there are p^2 algebraically independent absolute invariants of orders not exceeding p; (2) that these invariants are expressible in terms of certain p^2 geometric magnitudes; and (3) that consequently no relation common to all curves in the plane can exist among these geometric magnitudes. — The error in these statements is due to the fact that the differential form under consideration is of class zero, and hence has no gaussian invariants. There are therefore fewer than p^2 absolute invariants, and hence there exist relations among the p^2 geometric magnitudes mentioned. One such relation, of the third order, is readily established.

17. In Mr. Colpitts's paper, quintic curves lying on a hyperboloid were depicted upon a conic, and the bitangents, inflexions, etc., defined in terms of quartic involution of the quadrisecants. The rational form not lying on a quadric was discussed by means of the octic which fixes the points of contact of the stationary planes. The elliptic form was depicted into a plane curve of order 10, and the surface of trisecants was shown to belong to a linear complex. The form of genus 2 was also treated by means of the plane depiction. Finally, an exhaustive classification of the forms belonging to a linear complex was added.

18. In Mr. Breuke's paper the problem of finding the derivative of the function represented by a series of the form

$$(1) \qquad \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

was considered; and it was shown that if h is a positive integer and α a constant, and if the series

$$(2)\quad \sum_{1-h}^{\infty} \{(n-h)a_{n-h}\cos(nx-h\alpha)-(n+h)a_{n+h}\cos(nx+h\alpha) \\ +(n-h)b_{n-h}\sin(nx-h\alpha)-(n+h)b_{n+h}\sin(nx+h\alpha)\},$$

(where the a's and b's whose subscripts are not positive are all zero) converges uniformly throughout an interval which does not include or reach up to a root of $\sin h(x-\alpha)$, and if (1) converges at some point of this interval, and if as n becomes infinite $\lim a_n = \lim b_n = 0$, then (1) converges uniformly throughout the interval in question, and the function represented by (1) has at every point of this interval a finite derivative which is equal to the value of (2) divided by $2\sin h(x-\alpha)$. If $\alpha = 0$, $h = 1$, this reduces to a theorem of Lerch. Applications of the theorem to the summation of certain trigonometric series were indicated.

19. The purpose of Dr. Rabinovitch's paper is to lay down a set of necessary and sufficient kinetic axioms of geometry, and to prove their mutual independence and compatibility. The primary elements to start with are bodies and their places, and use is made of an auxiliary postulate of the continuity of time regarded as an aggregate of moments.

The axioms are 7 in number.

1) Divisibility.—Each body is divisible into two bodies.

2) Impenetrability.—Each body or each part occupies a place to the exclusion of other bodies or parts.

3) Motion.— A body can have different places at different times. Motion is proved to be a continuous process depending upon time.

By definition two bodies are contiguous when either can enter in part the region of the other during an infinitesimal interval of time, and a body is termed continuous when each of its parts is contiguous to some other parts of the same body.

Axioms 4) and 5) postulate the existence of continuous bodies termed rigid bodies or solids, *i. e.* such that, 1° any two contiguous parts remain contiguous throughout all their motions, and 2° to each particular place of any given part of such a body there corresponds a unique place for every other given part.

6) Homogeneity.—A part can be separated from a given solid, capable of being wholly placed within the space of any other given solid.

7) Axioms of completeness.—No motion is impossible unless it contradicts the above axioms.

From these are deduced the notions of surface, curve, and point. The places which can be occupied by the same solid, its surface or curve are termed congruent. The sphere, circle, straight line, plane, and angle are deduced from the notion of distance, which is a relation of all congruent couples of points. The socalled parallel postulate is deduced by proving that a certain continuous motion of a figure termed "immaterial quadrilateral" cannot contradict the above axioms.

F. N. COLE,
Secretary.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE nineteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Saturday, April 14, 1906. The total attendance was forty, including the following members of the Society:

Dr. L. D. Ames, Mr. G. D. Birkhoff, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Dr. E. L. Dodd, Mr. E. B. Escott, Dr. Peter Field, Professor G. W. Greenwood, Professor A. G. Hall, Professor E. R. Hedrick, Professor T. F. Holgate, Mr. Louis Ingold, Professor O. D. Kellogg, Dr. H. G. Keppel, Mr. N. J. Lennes, Professor H. Maschke, Professor E. H. Moore, Professor F. R. Moulton, Dr. L. T. Neikirk, Professor H. L. Rietz, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor H. E. Slaught, Professor E. J. Townsend, Dr. W. D. Westfall, Professor D. T. Wilson, Mr. R. E. Wilson, Professor Alexander Ziwet.

The chairman of the Section, Professor Alexander Ziwet, presided at the two sessions. Owing to the large number of papers on the programme, it was voted to reduce the time allotments by twenty-five per cent. It was also voted hereafter to restrict the printed programme to those papers for which titles and abstracts are in the hands of the Secretary on the date specified in the preliminary call for the meeting, and to request this notice to be made in the BULLETIN in connection with the announcement of meetings of the Section.

A report of progress was made by the Executive Committee concerning the proposal to devote one or more sessions of the December, 1906, meeting to the consideration of mathematics for engineers, and to invite representatives of engineering schools to present papers and to take part in the discussion.

A resolution was introduced by Professor E. H. Moore and unanimously carried, expressing the very earnest hope of the Chicago Section that it may be found possible to establish a strong section of the Society which shall hold meetings at some convenient center in the Southwest.

The following papers were read:

(1) Professor J. B. SHAW: "Significance of the term hypercomplex number."

(2) Professor J. B. SHAW: "Mathematical processes."

(3) Mr. LOUIS INGOLD: "A theorem on the intersections of Pascal lines."

(4) Professor H. F. BLICHFELDT: "On modular groups isomorphic with a given linear group."

(5) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups" (supplementary paper).

(6) Mr. A. R. SCHWEITZER: "On the foundations of abstract geometry" (preliminary report).

(7) Professor M. E. GRABER: "On the mathematical character of space intuition."

(8) Dr. J. C. MOREHEAD: "Note on the factors of Fermat's numbers."

(9) Mr. N. J. LENNES: "On functions of limited variation."

(10) Professor E. R. HEDRICK: "On the function $\xi(h)$ in the law of the mean."

(11) Professor F. R. MOULTON: "On the classes of periodic orbits computed by Professor G. H. Darwin."

(12) Mr. W. D. MACMILLAN: "On a certain type of periodic orbits in the problem of three bodies."

(13) Mr. F. L. GRIFFIN: "Certain periodic orbits of n finite bodies revolving about a relatively large central mass."

(14) Mr. W. R. LONGLEY: "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies."

(15) Professor H. L. RIETZ: "On normal correlation surfaces."

(16) Professor L. E. DICKSON: "Linear algebras in which division is always uniquely possible."

(17) Dr. E. L. DODD: "An application of Gibbs's exponential

dyadic function e^{ϕ} to the solution of linear vector and dyadic differential equations."

(18) Dr. E. L. DODD: "An example preparatory to the study of the ratio test for infinite series.

(19) Mr. R. E. WILSON: "On integral equations of the second kind."

(20) Mr. R. E. WILSON: "On integral equations of the first kind."

(21) Professor D. R. CURTISS: "A proof of the theorem concerning artificial singularities."

(22) Professor D. R. CURTISS: "On certain properties of wronskians and related matrices."

(23) Professor HEINRICH MASCHKE: "On spherical images of rectilinear congruences" (preliminary report).

(24) Dr. W. D. WESTFALL: "On integral equations of the second kind."

(25) Professor O. D. KELLOGG: "The roots of transcendental integral functions."

The papers of Professor Blichfeldt, Professor Graber, and Professor Maschke were read by title. Mr. MacMillan, Mr. Griffin, and Mr. Longley were introduced by Professor F. R. Moulton. Abstracts of the papers follow below, the numbers corresponding to the titles in the list above.

1. Professor Shaw's first paper compares critically the definitions of a hypercomplex number from the four standpoints of mutiplex, n-dimensional number, operator, and extension of ordinary number. The first definition is exemplified in the definition of the complex number $a + b\sqrt{-1}$ as a couple (a, b), where a and b are positive or negative reals, subject to the law of multiplication $(a, b)(c, d) = (ac - bd, ad + bc)$. The second is exemplified in the definition of $a + b\sqrt{-1}$ as a two dimensional number $ae_0 + be_1$, where $e_0^2 = e_0$, $e_1^2 = -e_0$, $e_0e_1 = e_1e_0 = e_1$. The third is exemplified in the definition of $a + b\sqrt{-1}$ as an operator which rotates through an angle $\tan^{-1} b/a$ and multiplies by $\sqrt{a^2 + b^2}$. The fourth defines $a + b\sqrt{-1}$ as a value of x which satisfies the ordinary equation $x^2 - 2ax + d = 0$, where $d = a^2 + b^2$. The latter algebraic definition is considered to be the most general since it does not require that from the equality of two hypercomplex numbers $a = b$ we must conclude $a_i = b_i$ $(i = 1, \cdots, n)$, where a_i is the i-th coordinate

of the number. This view permits us to call, for example, the algebra of positives and negatives a double or two-unit (qualitative unit) algebra, which the other definitions do not permit.

2. In his second paper Professor Shaw defined a mathematical process to be a definite procedure by which from given data we pass to a certain conclusion. The question of validity, objective or subjective, of the process is handed over to philosophy. The theory of mathematical processes bears to logic the relation that modern rational geometry has to euclidean geometry. It is also a generalized theory of mathematical forms. For example, on one hand the syllogism is exhibited as one of an infinite variety of modes of reasoning; on the other hand the theory of laws of combination, like the commutative law, is reduced to a problem in isomorphisms of substitution groups. This theory appears to be the common foundation of mathematics and logic. This paper is to be presented to the *Transactions* for publication.

3. In Mr. Ingold's paper are considered some properties of the configuration $\begin{pmatrix}12 & 3\\ 4 & 16\end{pmatrix}$ in their relation to a conic through six of its points. From these properties are immediately deducible a large number of results as to collinearity and concurrence of points and lines connected with the inscribed hexagon and inscribed (any) pentagon. In particular it is shown that certain joins (other than Pascal lines) of diagonal points of the hexagon meet by sixes in sixty points which lie by fours on the sides of the hexagon and by threes on twenty other lines. The results applied to a reciprocal situation show that certain of the intersections of Pascal lines (Cayley's 360 points, *Quarterly Journal*, volume 9) lie by sixes on sixty lines which pass by tens through the vertices of the hexagon.

4. In his first paper Professor Blichfeldt proves the following theorem: Given a group G of linear homogeneous substitutions in n variables, transitive (irreducible) and of finite order, then there exists an infinitude of prime numbers p for which we can construct an isomorphic transitive group G' of linear homogeneous substitutions in n variables, the elements of whose matrices are integers taken modulo p.

5. Professor Blichfeldt's paper on the order of linear homogeneous groups is supplementary to his two former papers in

the *Transactions*. He reduces his former superior limit $(n-1)(2n+1)$ of the value of a prime p which may divide the order of a finite primitive group of linear homogeneous substitutions of determinant unity in n variables. If such a group has a substitution of order p $(p > n)$ and variety m $(m \leqq n)$, but none of order pk, then $p \leqq 3m + n - 3$. The primes which may divide the orders of the primitive collineation groups in 4, 5 and 6 variables are not greater than 11, 13 and 19, respectively. The two papers by Professor Blichfeldt will appear in the July number of the *Transactions*.

6. Mr. Schweitzer sets up systems of axioms for n dimensions $(n = 1, 2, 3, \cdots)$. Two undefined symbols enter the theory for n dimensions, viz., points as elements and a relation K, effective under conditions specified in the postulates, between two ordered n-ads of points. The postulates are laid down so as to permit of ready definition of betweenness, collinearity, etc. Systems III, etc., are shown to be sufficient for projective geometry. System I is easily extensible to higher dimensions on comparison with systems II, III, $\cdots$. The extensions of the remaining systems are quite obvious. Such extensions are, however, less elegant than the corresponding sets in the first systems.

7. In this paper Professor Graber develops the theory that space intuitions do not change in nature, but in number and complexity. That in the naïve intuition of space the intuitional elements are few and their complexes restricted within narrow limits, but as geometric knowledge increases, the number of intuitions and intuitional complexes increases, and we have consequently more exact knowledge of spatial relations.

In the intuitional complexes formed by combinations of intuitions a, b, n, viz., $\Sigma(a, b, c, \cdots, n)$, we may have (1) annihilators, *e.g.*, $\widehat{ab} \equiv 0$; (2) correctors, *e.g.*, $\widehat{ab} = a, b$, or a complex partaking of the nature of a and b; (3) transmutants, *e.g.*, $\widehat{ab} =$ a complex indicating a distinct advance.

It is then attempted to show that the ensemble of our space intuitions sustains a functional relationship to geometric progress, the intuitions leading to geometric research and vice versa. From this it would follow that no geometric problem is completely solved until we are able to interpret both the method of solution and the results in terms of the intuition.

8. Dr. Morehead's paper appears in full in the present number of the BULLETIN.

9. The variation of a function $f(x)$ on an interval ab is denoted by $V_a^b f(x)$. Mr. Lennes's paper is concerned with the variation of the definite integral regarded as a function of its limits of integration. It is known that $\int_a^x f(x)dx$ is a function of limited variation on an interval ab on which $f(x)$ is properly integrable. It is shown that

$$V_a^b \int_a^x f(x)dx = V_a^b \int_a^x |f(x)|\,dx.$$

In case the integral $\int_a^b f(x)dx$ exists improperly it is shown that the existence of $\int_a^b |f(x)|\,dx$ is a necessary and sufficient condition that $\int_a^x f(x)dx$ shall be a function of limited variation on ab. A new proof is given of a theorem proved by Jordan and Stolz, and announced by Moore with reference to these authors (*Transactions*, volume 2, page 322), viz., that the existence of the (Moore's) broad integral is sufficient for the existence of $\int_a^b |f(x)|\,dx$ either narrowly or broadly.

10. In this paper Professor Hedrick develops some of the properties of the function $\xi(h)$ in the law of the mean, written in the form

$$\frac{f(a+h)-f(a)}{h} = f'(a+\xi).$$

In particular it is shown that this law holds in a slightly modified form even in case the derivative $f'(x)$ exists merely with regard to certain assemblages. The possible discontinuities of $\xi(h)$ and of $f'(x)$ are then discussed. It is shown that $\xi(h)$ is always continuous at $h=0$ with respect to an assemblage which has the power of the continuum in any interval about $h=0$; and again that $f'(x)$ is not necessarily continuous for the entire assemblage of even those values which ξ actually assumes.

Finally, remarks concerning the general nature of a derivative, and concerning the necessary and sufficient conditions for a minimum of $f(x)$ close the paper. The paper will be presented to the *Annals of Mathematics*.

11. The paper of Professor Moulton consists of an application of the analysis which he presented to the Society at the summer meeting, 1905, to the periodic orbits discussed by Professor G. H. Darwin in *Acta Mathematica*, volume 21. The satellites of class A are of the type of the orbit of the moon when its inclination and eccentricity are neglected. A numerical example agreeing with Darwin's figures is given. The orbits of classes B and C are the analytic continuations of orbits whose coordinates are conjugate complex numbers for vanishing values of the disturbance. When expanded as power series in the lunar theory parameter m, the expressions for the coordinates of the orbits A, B and C all have the same radius of convergence. These series cease to converge for the value of m for which B and C become real; and this value of m probably gives their true radius of convergence. Numerical application is also made to planets A. The oscillating satellites were treated in an earlier paper.

12. Mr. MacMillan treats in his paper a certain type of periodic orbits of an infinitesimal body under the attraction of two finite bodies which revolve about one another in ellipses. The existence of periodic orbits of this type in the plane is demonstrated and a convenient method is given for constructing the solutions as power series in a parameter with periodic coefficients.

13. In this paper Mr. Griffin discusses a certain class of periodic solutions of the problem of n bodies where one of the masses is very large with respect to the others. In other words the distribution of the masses is such as is presented by the sun and any number of planets, or by a planet and any number of satellites. The bodies move in a plane, with their undisturbed synodic mean motions commensurable. Using the differential equations in all their generality, the existence of periodic orbits under the mutual disturbances is proved, the demonstration being particularly simple when all the bodies start from a "symmetric conjunction." A method of constructing the solutions as power series in the ratio of one of the

small masses to the large one is given in detail, the solution being unique when the period, the epoch, and the longitude of the initial conjunction have been selected. As an application, numerical work is given for the case of Jupiter's satellites, I, II, III. The fact that under certain circumstances there are no periodic orbits seems to offer information as to the reason for the existence of the "lacunary spaces" in the system of asteroids, and in Saturn's rings.

14. Mr. Longley assumes that the coordinates of the n finite bodies are known periodic functions of the time; for example, the generalized lagrangian solutions. The orbits considered are those of a particle of infinitesimal mass which revolves about one of the finite bodies. The problem is restricted to one plane, and the differential equations of motion are treated rigorously, the solutions being expressed as power series in parameters which converge when they are sufficiently small. It is shown that when a period (necessarily a multiple of the period of the finite bodies) is preassigned, there exist two and only two orbits of the particle, of the type considered, which are periodic with the prescribed period. The motion in one is direct and in the other retrograde. A convenient method for constructing the solutions is given, and some special cases are considered in detail.

15. Professor Rietz gives a derivation of correlation surfaces based upon the nature of certain plane sections of the surface. In the derivations hitherto given, the form of the function which defines the surface has either been assumed, or justified by assumptions as to the distribution and independence of the causes which produce deviations from the mean value. Then the constants in this function are determined. Much use has been made of correlation surfaces in recent years for the purpose of giving mathematical expression to facts of organic evolution. From the manner in which the applications are made, the nature of plane sections parallel to the two coordinate planes is of primary importance. It therefore seems well to make this the starting point of a derivation of these surfaces. It is shown in this paper how correlation surfaces which describe many natural phenomena can be obtained from certain types of sections.

16. Professor Dickson considers algebras whose elements are

$\Sigma a_i e_i$, the m coördinates a_i ranging over an arbitrarily given field F. For $m=3$, he considers the most general linear transformation of the units and exhibits families of algebras invariant under every linear transformation. The algebras which admit more than one transformation (and hence exactly three transformations) into themselves are completely determined. From each of these standpoints he is led to the same remarkable set of families of algebras, each set characterized by a parameter μ. For $\mu=1$, the family is the system of all fields of rank 3 with respect to F. For $\mu=0$, the family is the system of algebras

$$i^2=j, \quad ij=ji=b+\beta i+Bj, \quad j^2=4bB-\beta^2-8bi-2\beta j,$$

where b, β, B range over the marks of F (assumed not to have modulus 2) for which $x^3-b-\beta x-Bx^2$ is irreducible; in such an algebra division is always uniquely possible. For m an even integer $\geqq 4$ and F an arbitrary field, there exists a very interesting type of algebra in which division is always possible. For $m=4$, the algebra is given by

$$\begin{array}{c} i^2=j, \quad ij=ji=k, \quad ik=ki=c, \quad j^2=-c+dj, \\ jk=kj=-ci+dk, \quad k^2=cd-cj \end{array} \quad \left(\begin{array}{c} c \text{ and } d^2-4c \\ \text{not squares in } F \end{array}\right).$$

In fact, its determinant equals

$$(x^2+dxy+cy^2)^2-c(z^2+dzw+cw^2)^2,$$

which vanishes only when x, y, z, w all vanish.

While there exists a large number of non-equivalent m-tuple algebras in a given field (for example, if $m=3$ and F is the field of order 3 or 5, the number is 6 or 38 respectively), there appears to be a single non-field commutative algebra for each field when $m=3$ or 4. However, for $m=6$ there exist two non-equivalent, non-field commutative algebras in every field.

A general investigation was made of linear associative algebras in which division is always uniquely possible. An interesting example of such an algebra is one with the n^2 units $i_0^r e^s$ $(r, s=0, 1, \cdots, n-1)$, where $e^n=\tau=$ rational function of i_0, and $e^s i_0^r=i_s^r e^s$, while i_0, $i_1=\theta(i_0)$, $i_2=\theta^2(i_0)$, $\cdots$ are the roots of a uniserial abelian equation of degree n irreducible in F. Further, the coefficients of the latter and τ must satisfy certain conditions which can be expressed in terms of the theory of homogeneous forms of degree n. The paper will appear in the *Transactions*.

17. Dr. Dodd's first paper is in abstract as follows: Gibbs defines the direct product $\alpha \cdot \beta$ of two vectors as the product of their lengths into the cosine of the included angle. A dyad $\gamma\delta$ is an operator transforming one vector into another. Thus $(\gamma\delta) \cdot \alpha = (\delta \cdot \alpha)\gamma$. The product of two dyads is a dyad, *e. g.*, $(\gamma\delta) \cdot (\epsilon\zeta) = (\delta \cdot \epsilon)(\gamma\zeta)$. A dyadic, Φ, is a sum of dyads, and may be expressed in nonion form, thus:

$$\Phi = a_{11}ii + a_{12}ij + a_{13}ik + a_{21}ji + a_{22}jj + a_{23}jk + a_{31}ki + a_{32}kj + a_{33}kk,$$

where i, j, k, are unit vectors at right angles, and the a's are scalars. In the product, $\Phi \cdot \Psi$, multiplication is distributive and, if Φ and Ψ are in nonion form, follows the row-column rule for the multiplication of determinants. $\Phi^2 \equiv \Phi \cdot \Phi$. The idem-factor $I = ii + jj + kk$ is such that $I \cdot \rho = \rho$ and $\rho \cdot I = \rho$, where ρ is the vector. Finally Gibbs defines

$$e^{\Phi} = I + \Phi + \frac{1}{2!}\Phi^2 + \frac{1}{3!}\Phi^3 + \cdots$$

We find that, in general, $e^{\Phi+\Psi} \neq e^{\Phi} \cdot e^{\Psi}$, but if t and Δt are scalars, *i. e.*, numbers, real or complex, then $e^{(t+\Delta t)\Phi} = e^{t\Phi} \cdot e^{\Delta t\Phi}$; so that

$$\frac{d}{dt}e^{t\Phi} = \Phi \cdot e^{t\Phi} = e^{t\Phi} \cdot \Phi.$$

Then, if ρ is a vector and $\Gamma_0, \Gamma_1, \cdots, \Gamma_n$ constant dyadics, a solution of

$$(1) \qquad \Gamma_n \cdot \frac{d^n\rho}{dt^n} + \Gamma_{n-1} \cdot \frac{d^{n-1}\rho}{dt^{n-1}} + \cdots + \Gamma_1 \cdot \frac{d\rho}{dt} + \Gamma_0 \cdot \rho = 0$$

will be $\rho = e^{t\Phi} \cdot \alpha$, if α is an arbitrary vector, and Φ is a solution of

$$\Gamma_n \cdot \Phi^n + \Gamma_{n-1} \cdot \Phi^{n-1} + \cdots + \Gamma_1 \Phi + \Gamma_0 = 0.$$

In (1), ρ may be replaced by a dyadic Ψ if a dyadic Ω replaces α. If in (1) $\Gamma_n = c_n I$, etc., where the c's are scalar constants, then (1) reduces to

$$c_n\frac{d^n\rho}{dt^n} + c_{n-1}\frac{d^{n-1}\rho}{dt^{n-1}} + \cdots + c_1\frac{d\rho}{dt} + c_0 =$$

Here substitute $\rho = e^{mtU}\cdot\alpha = e^{mt}\alpha$, where m is a scalar. Cf. Grassmann's Ausdehnungslehre, § 498.

18. The example of ratio test devised by Dr. Dodd is the following:

Let $$S = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

$$= \frac{1}{5} + \frac{1\cdot 3}{5\cdot 10} + \frac{1\cdot 3\cdot 5}{5\cdot 10\cdot 15} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{5\cdot 10\cdots 5n} + \cdots.$$

Then $$r = \frac{u_{n+1}}{u_n} = \frac{2n+1}{5(n+1)}, \quad \lim_{n=\infty} r = \frac{2}{5}, \quad r < \frac{2}{5}.$$

The sum of S exists, and is less than 0.2912, by comparison with

$$S' = u_1 + u_2 + u_3 + u_4 + u_5 + (2/5)u_5 + (2/5)^2u_5 + (2/5)^3u_5 + \cdots.$$

The sum of S is greater than .2909, by comparison with

$$S'' = u_1 + u_2 + u_3 + u_4 + u_5 + (11/30)u_5 + (11/30)^2u_5 + \cdots.$$

Then $S = 0.291$ is correct to within 0.0002.

An infinite series whose test ratio approaches a limit l will ultimately become approximately a geometric series with ratio l, and it will converge if $|l| < 1$.

19. Mr. Wilson's first paper treats of the reduction of a differential equation of the third order under certain boundary conditions to an integral equation of the second kind, and a generalization to differential equations of order $2n - 1$. After showing that the matrix of such an integral equation is always skew-symmetric, he applies his results to the differential equation $y''' - 4py' - 2p'y = 0$, which arises in the solution of Lamé's equation $y'' = (Ax + B)y$ by the application of a method due to Hermite, and shows that for a fixed value of A there exists an infinity of normal solutions which satisfy the given boundary

conditions. He further shows under what conditions an arbitrary function is developable in terms of these normal solutions.

20. In his second paper Mr. Wilson discusses the representation of arbitrary functions by means of Fourier's double integrals and shows the relation of the theory of integral equations of the first kind to the theory of Fourier's integrals. Incidentally he gives a few examples of integral equations of the first kind which are readily solved and which in themselves might serve as the basis of Fourier's theory.

21. The first paper of Professor Curtiss gives a proof of the fundamental theorem which states that if $f(z)$ is single valued and analytic at all points in the neighborhood of a point c, exclusive of c, and remains finite throughout this neighborhood, it has at most an artificial singularity at c; *i. e.*, with a suitable definition of $f(c)$ the function $f(z)$ will be analytic at the point c. In some fallacious proofs use has been made of the auxiliary function

$$\phi(z) = (z - c)f(z) \quad (z \neq c),\ \phi(c) = 0.$$

A very simple demonstration is here given, based on the consideration of the function $(z - c)\phi(z)$. This paper will be presented to the *Annals of Mathematics* for publication.

22. The second paper of Professor Curtiss will appear in the July number of the BULLETIN.

23. Professor Maschke discussed the solution of the general problem of the determination of rectilinear congruences whose spherical images are given, *i. e.*, the determination of x, y, z in terms of X, Y, Z and the quantities e, f, f', g defined by

$$\Sigma \frac{\partial X}{\partial u}\frac{\partial x}{\partial u} = e, \quad \Sigma \frac{\partial X}{\partial u}\frac{\partial x}{\partial v} = f, \quad \Sigma \frac{\partial X}{\partial v}\frac{\partial x}{\partial u} = f', \quad \Sigma \frac{\partial X}{\partial v}\frac{\partial x}{\partial v} = g.$$

24. In this paper Dr. Westfall shows that the restriction that the matrix $K(s, t)$ be symmetric in s, t in Hilbert's treatment of the integral equation of the second kind

$$f(s) = \phi(s) - \lambda \int_0^1 K(s, t)\phi(t)dt$$

is not necessary for that treatment. A slight change in his proof gives the relations

$$K(s, t) + \mathrm{K}(s, t) = \lambda \int_0^1 K(s, r)\mathrm{K}(r, t)dr,$$

$$K(s, t) + \mathrm{K}(s, t) = \lambda \int_0^1 \mathrm{K}(s, r)K(r, t)dr,$$

which prove the existence and uniqueness of the solution

$$\phi(s) = f(s) - \lambda \int_0^1 K(s, t)f(t)dt.$$

25. The fact that the roots of an integral algebraic function are continuous functions of the coefficients may be generalized to transcendental functions, and the result very simply applied to give certain information concerning the roots of the latter. Professor Kellogg proposes two applications of this notion, the first in building up a transcendental integral function term by term, so that it appears that if the convergence of the series is rapid enough, it will surely have finite roots. By a second application the series is considered as a polynomial plus a remainder. If the remainder is sufficiently small all the roots of the polynomial have corresponding roots in the complete function.

H. E. Slaught,
Secretary of the Section.

Chicago, Ill.,
April 20, 1906.

GROUPS IN WHICH ALL THE OPERATORS ARE CONTAINED IN A SERIES OF SUBGROUPS SUCH THAT ANY TWO HAVE ONLY IDENTITY IN COMMON.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, April 28, 1906.)

1. We begin with the case where the group G is any abelian group such that all of its operators are contained in a series of subgroups $H_1, H_2, \cdots, H_\lambda$ any two of which have only

identity in common. In other words, the λ given subgroups contain every operator of G, except identity, once and only once. It is easy to prove that the order of G is p^m, where p is a prime, and that the type of G is $(1, 1, 1, \cdots)$. If the order of G were divisible by two distinct primes, one of the λ subgroups H_α would involve an operator of some prime order p while another subgroup H_β would involve an operator of a different prime order q. The product of these two operators would be in H_γ, where γ is different from α and β. As the qth power of this product would be in H_α, the two subgroups H_α, H_γ would have an operator of order p in common. Since this is contrary to the hypothesis, it follows that the order of G is p^m.

If one of the λ subgroups H_α contained an operator of order p^2, the product of this operator into an operator of order p from H_β would be in H_γ. As this operator would be of order p^2 and would have its pth power in H_α, it follows that none of the subgroups in question can involve an operator of order p^2 and hence G is of type $(1, 1, 1, \cdots)$. Moreover, whenever G is such an abelian group, its operators may be arranged in subgroups having only identity in common. One such arrangement is effected by letting each of the H's represent a subgroup of order p. In this case $\lambda = (p^m - 1)/(p-1)$.

In what precedes no condition was imposed upon the subgroups $H_1, H_2, \cdots, H_\lambda$ except that they include all the operators of G and that any two of them have only identity in common. If we impose the additional condition that all of these subgroups are of the same order, it follows from the preceding paragraphs that this order is a power of a prime. Moreover if each of these subgroups is of order p^a, $\lambda = (p^m-1)/(p^a - 1)$. That is, m is divisible by a. That this condition is sufficient as well as necessary follows from the fact that we may represent G as the direct product of m/a subgroups of order p^a. If these subgroups are represented as regular substitution groups, the λ subgroups may be obtained by arranging these substitution groups in (1, 1) correspondence (taking first two at a time, then three at a time, etc.) and then transforming cyclically $p^a - 1$ elements of a constituent which differ from the identity.*

The main results which have been obtained in the preceding paragraphs may be stated as follows: All the operators of an abelian group cannot be arranged in subgroups such that any two have only identity in common unless the order of the group is p^m and its type is $(1, 1, 1, \cdots)$. The necessary and sufficient condition that all the operators of this abelian group can be arranged in subgroups of the same order p^a, such that any two of these subgroups have only identity in common, is that m is a multiple of a.

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* Cf. Moore, BULLETIN, vol. 2 (1895), p. 38.

2. We shall now consider the case where the group G is non-abelian and of order p^m.

As in the preceding case we do not, at first, impose any conditions upon the orders of $H_1, H_2, \cdots, H_\lambda$. Suppose that at least one of these subgroups H_a contains operators of order p^2. Since every subgroup of a group of order p^m is transformed into itself by operators which are not in it, H_a is invariant under a group of order p^{a+1}, p^a being the order of H_a. This contains no operators of order p^2 except those of H_a, since no two of the H's have operators of order p in common. The operators of order p^2 in H_a generate a characteristic subgroup K under this group of order p^{a+1}. As the latter is invariant under a group of order p^{a+2}, K must also have this property. If this group of order p^{a+2} involved any operators of order p^2 besides those of K, such an operator into an invariant operator of order p in K would give a product of order p^2 which would be in a different H but would generate the same subgroup of order p as the preceding operator of order p^2. As this is contrary to the hypothesis that any two H's have only identity in common, and as the same argument would apply to larger groups if the group of order p^{a+2} did not involve any operators of order p^2 besides those of K, we have proved the theorem: *If a non-abelian group of order p^m is such that all of its operators are found in a series of subgroups of which no two have any common operator except identity, then only one of these subgroups can involve operators of order p^2.*

We impose now the additional condition that the subgroups $H_1, H_2, \cdots, H_\lambda$ have the same order and prove that, in this case, all the operators of the non-abelian group of order p^m are of order p, with the exception of identity. If this were not the case, the operators of order p^2 would generate a characteristic subgroup K contained in H_a. Any other H would transform K into itself and hence it would involve operators of order p which would be commutative with operators of order

p^2 in K. As the product of such operators of order p and p^2 respectively would be of order p^2 but would not be in K, this is impossible. That is, *G cannot involve any operators of order p^2 when the subgroups $H_1, H_2, \cdots, H_\lambda$ have the same order.* In fact, the preceding proof holds when the order of the largest of these subgroups does not exceed p times the order of some other one of them.

The preceding proof can be directly extended so as to apply to any group of any order whatsoever in which all the operators are found in a series of subgroups of the same order such that any two of them have only identity in common. That is, such a group G cannot involve any operator whose order is the square of some number.* Suppose that G involved an operator of order p^2, where p is a prime, and let P represent one of its Sylow subgroups of order p^m. If $P_1, P_2, \cdots, P_\lambda$ represent the subgroups of P which are found in the different subgroups of G which have only identity in common, it follows from what was proved above that not more than one of these subgroups can involve operators of order p^2. The operators whose orders exceed p in P would therefore generate a subgroup of order p^α where α does not exceed $\frac{1}{2}m$. As this is impossible, we have proved that G cannot contain an operator whose order is a square greater than unity.

NOTE ON THE FACTORS OF FERMAT'S NUMBERS.

BY DR. J. C. MOREHEAD.

(Read before the Chicago Section of the American Mathematical Society, April 14, 1906.)

FERMAT'S numbers $F_n = 2^{2^n} + 1$ are known to be prime for $n = 0, 1, 2, 3, 4$, and composite for $n = 5, 6, 7, 9, 11, 12, 18, 23, 36, 38$. By calculating the residues (mod $2^{75}\cdot 5 + 1$) of the reciprocals †

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* The minimum order of G is evidently the square of the order of one of these subgroups. Dr. Manning proved that G is abelian whenever it has this minimum order.

† In many cases the residue of $1/2^{2^n}$ (mod N) is more readily calculated than the residue of 2^{2^n}. In the present case $-2^{75}\cdot 5 \equiv 1 \bmod (2^{75}\cdot 5 + 1)$. Therefore $1/2^{2^6} \equiv -2^{11}\cdot 5$, $1/2^{2^7} \equiv 2^{22}\cdot 5^2$, $\cdots$, $1/2^{2^9} \equiv 2^{88}\, 5^8 \equiv -2^{13}\cdot 5^7$, $\cdots$, $1/2^{2^{12}} \equiv -5^{26}\cdot 10^{29}$, at which stage division by $2^{75}\cdot 5 + 1$ may be begun.

$$1/2^{2^{26}}, \quad 1/2^{2^{27}}, \quad 1/2^{2^{28}}, \cdots$$

I have found that

$$-1 \equiv 1/2^{2^{73}} \pmod{2^{75}\cdot 5 + 1},$$

which establishes the composition of F_{73}.

This leads to an interesting secondary result,—the identification of the *twenty-four place* prime*

$$P = 2^{75}\cdot 5 + 1 = 188\ 894\ 659\ 314\ 785\ 808\ 547\ 841.$$

The prime character of P is a consequence of the fact that all factors of F_{73} must have the form

$$Q = 2^{75}q + 1.$$

For, if P were not prime then a factor of P, $< \sqrt{P}$, and hence not of the form Q, would be a factor of F_{73}.

Further computation recently carried out shows that under $2^{79}\cdot 5 + 1$, the only factors of F_n of the form $2^{\kappa}\cdot 5 + 1$ are

$$2^{7}\cdot 5 + 1, \quad 2^{25}\cdot 5 + 1, \quad 2^{39}\cdot 5 + 1, \quad 2^{75}\cdot 5 + 1;$$

and in fact, for $\kappa < 79$, $2^{\kappa}\cdot 5 + 1$ is prime only for $\kappa = 1, 3, 7, 13, 15, 25, 39, 55, 75$.

An investigation of the numbers $2^{\kappa}\cdot 3 + 1$ shows that $2^{41}\cdot 3 + 1$ is the only number of this form under $2^{105}\cdot 3 + 1$ that is a factor of a Fermat number, the only *odd* values of κ for which $2^{\kappa}\cdot 3 + 1$ is prime being $\kappa = 1, 5, 41$. A prime of the form

$$\Pi = 2^{2\kappa}\cdot 3 + 1$$

cannot be a factor of a Fermat number. For if Π is a factor of F_n, $n \leqq 2\kappa - 2$, the congruence

$$2^{(\Pi-1)/3} \equiv 1 \pmod{\Pi} \tag{1}$$

must be satisfied; and this requires that Π be expressible in the form †

$$\Pi = a^2 + 3b^2 \tag{2}$$

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* $2^{61} - 1$ is the highest number hitherto definitely known to be prime. See Seelhoff, *Zeitschrift für Math. u. Phys.*, vol. 31 (1886), p. 178; and Ball, *Messenger*, vol. 21, pp. 34–40.

† For general conditions that $2^{(p-1)/n} \equiv 0 \pmod{p}$, see Cunningham, *Quar. Jour. of Math.*, vol. 37 (1905), pp. 124–135.

where

$$b \equiv 0 \pmod{3}. \tag{3}$$

Since a given prime is expressible in not more than one way in the form $a^2 + 3b^2$, and since

$$\Pi = (1)^2 + 3(2^\kappa)^2$$

it follows that condition (3), and therefore (1), is not satisfied.

It is easy to show that *composite* numbers of the forms $2^\kappa \cdot 3 + 1$, $2^\kappa \cdot 5 + 1$ can not be factors of Fermat's numbers.

NORTHWESTERN UNIVERSITY,
April, 1906.

THEORETICAL MECHANICS.

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies. BY E. T. WHITTAKER. Cambridge, The University Press; New York, The Macmillan Company, 1904. xiii+414 pp.

AT Cambridge, England, mathematics means for the most part mechanics, mathematical physics, or even physics sometimes not so very mathematical. The famous tripos — the mathematical tripos, of course, which goes back at any rate to 1747—seems, at least to an outsider, to lay its main stress on the theoretical application of mathematics rather than on pure mathematics. Very likely this is a tradition that has come down from the time of Newton, and it is certainly maintained by the eminent physicists such as Stokes, Kelvin, Maxwell, Rayleigh, J. J. Thomson, who have been high wranglers. This tripos with its great prestige gives an attractive and distinctive touch to the university and although the ever-increasing pressure of pure mathematics, with its possibilities for various kinds of unessential and mediocre work infinitely wider than those to be found in theoretical applied mathematics, will probably tell on the training at Cambridge sooner or later, may we not look forward to that date with some regrets on the general uniformizing that is taking place and lament the fact that other realms than physics are possessed of an entropy? At present, however, there seems no immediate danger. The one year of 1904 saw the publication at the Cambridge University Press of three large and highly important volumes by as many recent graduates of Cambridge typically cantabrigian in that they exhibit great mathematical power and attainments directed firmly and unerringly along the direction of physical research. It is needless to say that we refer to Walker's Analytical theory of light, Jean's Dynamical theory of gases, and Whittaker's Analytical dynamics.

Mechanics is more largely pure mathematics than the other branches of theoretical physics because the data are fewer and simpler; the transition to the systems of differential equations, less recondite. And so far is mechanics developed that probably it is the mathematician who must carry on the work rather than the physicist engrossed in advancing the experimental data of science—witness the recent workers, Poincaré, Painlevé, Hadamard, Duhem, Volterra, Levi-Cività, Gibbs, Love, Jeans, Klein, and others. For the mathematician, however, the way has unfortunately been barricaded by the kind of treatise on mechanics which has been available to the intermediate student. The famous works of Routh are not so much mathematical as distinctly mechanical; the reader learns rather how to do difficult and often artificial problems rather than how problems are done. This training is highly valuable and should precede other work: but where to find the other? It is precisely at this point that Whittaker breaks the barricade and opens the way to fruitful advance. His book, to be sure, starts at the beginning of the subject—but it is not for the beginner. A good course from the practical problem-solving aspect of Routh is almost indispensable to a ready and successful mastery of Whittaker. The book is mathematical in nature, written with a precision and developed with a logic sure to appeal to mathematicians. It is modern, thoroughly modern with its bibliographical references running quite up into the year 1904. Its subjects include the contributions of "Lie, Rayleigh, Klein, Hertz, Lorentz, Poincaré, Siacci, Bruns, Boltzmann, Larmor, Greenhill, Appell, Painlevé, Stäckel and Levi-Civita" to quote from the preface. And this is quite true. Those who are familiar with a number of treatises on mechanics may scan them in vain for a résumé of the methods introduced since 1880 just as they may look equally in vain for an advanced book arranged in what seems a logical order from the mathematician's point of view.

It is therefore a definite and strategic position which Whittaker has occupied and one may in this instance put aside all lamentations from the observation that of the making of books there is no end. To show this, needs but a detailed examination of the contents. First, however, it may be well to mention that the methods of attack which the author displays throughout the text are numerous. Be they geometric or algebraic or analytic he invariably selects that which moves most directly to the goal of solution. This diversity of method taken with the compact style makes the book hard reading for any but the somewhat advanced student. To solve the exercises requires on the part of the reader a similar versatility. Thus one may see why and how it is that the Cambridge student well versed in mechanics is also in reality thoroughly equipped in pure mathematics in so far as present practical applications are concerned. To a greater extent than his predecessors Whittaker has availed himself of the great machinery of modern mathematics, has thrown life into both the mathematics and the mechanics which he treats, and in this way has produced a book which may well become an inspiration to the next generation as Routh's treatises have been to this.

The first chapter of the text is on kinematic problems. Here, condensed into twenty-five pages, may be found Euler's, Hamilton's and Chasles's theorems on displacements, with their applications to instantaneous axes and centers; also the various methods of representing a displacement analytically, whether by the angle of version, the coördinates of Rodrigues, the angles of Euler, or the parameters of Klein, with the formulas for transformation from one to another and the expressions for the velocity and accelerations in the different systems of coördinates. It should be stated that the chapter further contains about twenty-five problems of all sorts and grades for the reader to work. This is a feature of the book. The problems cannot all be equally well selected and equally instructive. A large number of them are branded "(Coll. Exam.)" and frequently they would have shown it only too obviously without the mark. On the other hand many are taken from published memoirs of prominent writers and are excellent, though often difficult. The solution of a considerable number of problems chosen throughout the work indicates that the answers are usually correct. The second chapter is on the equations of motion and is of equal length—or rather brevity. It does for dynamics what the first did for kinematics. From the simple newtonian equations for the motion of a particle the author proceeds by rapid strides to the ideas of work and of generalized coördinates, to the distinction between holonomic and non-holonomic systems, and to the lagrangian equations for holonomic systems, to conservative systems, kinetic potential, quasi coördinates, and finally to impulsive motion. It is needless to point out that for one not already somewhat familiar with the subject these twenty-five pages must be taken in small doses mixed with similar portions taken from the fourth and later chapters. This second chapter is purely on the theory and not on the practice of the equations of dynamics.

In the discussion of the principles available for integration, which makes up Chapter III, the author continues his mathematical investigations in much the same style. The reader will again do well to intersperse some parts of the later chapters with his reading. With this and the ten or a dozen problems given in immediate connection with the investigations there is no very great difficulty to impede one's progress. After a word about what is meant by the solution of a system of differential equations, the question of ignorable coördinates is taken up in detail, and the theorem is proved that a dynamical system with n degrees of freedom which has k ignorable coördinates can be reduced to a dynamical system which has only $n - k$ degrees of freedom. And this possibility of ignoring some coördinates turns out to be the most general reason for the actual solvability of some problems and non-solvability of others. Too much emphasis cannot be laid upon this principle of the ignoration of coördinates, for it gives a definite and regular method of simplifying the problem, and frequently it alone may be made to yield the desired information about a system the actual complete solution of which we are either unable to obtain or able to obtain only by the expenditure of too much effort. Simple examples of the principle are found in systems possessing one or more integrals of momentum or of angular momentum. In this connection it should be noted that the kinetic energy of the ignored coördinates appears as potential energy in the modified system. This has given rise to a question which we are not yet in a position to answer satisfactorily, namely whether or not all potential energy may be explained as kinetic energy of some ignored coördinates, and how? It is obvious to all, that potential energy is to a considerable extent merely a convenient name for the manifestations of something the mechanism of which we do not understand and frequently "explain" by action at a distance. This instructive chapter closes with a discussion of the integral of energy in a conservative system and its use in reducing by one the number of degrees of freedom of the dynamical system.

The forty-five pages of Chapter IV on the solvable problems of particle dynamics form, when taken with the general theory of the preceding chapters, a veritable treatise on dynamics of a particle. From the simplest case of the pendulum the author goes through the various forms of central motion to motion on different kinds of surfaces. So many of the problems are soluble in finite form only in terms of the elliptic functions that some knowledge of these functions alike in the legendrian, the jacobian, and the weierstrassian analysis is indispensable. Fortunately the author has his own excellent work on modern analysis on which to fall back,* and references to it are frequent In this chapter there are about seventy problems, many of which should be interspersed with Chapters II and III, purely for pedagogic purposes. In fact, although the arrangement in the text is admirable for theoretical and logical purposes, instruction requires the interlarding of one part with another. There follows a short chapter of fifteen pages, with about as many problems, on the dynamical specification of bodies. It contains the usual theorems on moments, products, and ellipsoids of inertia, on kinetic energy, and on the independence of the motion of the center of gravity and of the motion relative to it. This leads directly to Chapter VI on the soluble problems of rigid dynamics — a complete analogue to Chapter IV as regards length, number of problems, and so forth. Naturally the top comes in for a lion's share of the space, and the elliptic functions are as indispensable as before.

Up to this point, about two fifths of the whole work has been covered with very little of a really advanced mathematical character except perhaps the technique of the elliptic functions. From here on, however, the amount of mathematics which the reader must know, or assimilate from the meager indications in the text, is by no means inconsiderable — calculus of variations, elementary divisors, integral invariants, and contact transformations may be mentioned as examples. It is, then, just at this point that modern mechanics begins.

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* E. T. Whittaker, A Course in Modern Analysis, reviewed by Bôcher in the BULLETIN, vol. 10, pp. 351–354.

The seventh chapter of some thirty-five pages deals with the theory of vibrations whether about a state of equilibrium or of motion. The present method of treatment goes back to Weierstrass and requires a tolerable knowledge of the theory of quadratic forms, linear transformations, and elementary divisors. At times the analysis itself, apart from what it presupposes, is somewhat complicated. Nevertheless a good student even poorly equipped ought to get all the chief points with the aid of the forty or so problems which he may solve in connection with the text. The more difficult of these are a task for the best; but the easier are very simple and illustrate the principles about as well. It is remarkable how much mathematics that one does not fully understand will cheerfully be accepted on faith if only a number of problems come out successfully when the indicated method is applied—and this is often the best way to get to understand the mathematics.

Chapter VIII on non-holonomic systems and dissipative systems is admirable and admirably placed for theoretical purposes. In elementary and practical treatments of mechanics it is necessary to introduce such systems earlier because so many of the simplest systems are of this type; but the difficulties in the proper theoretical treatment of these systems render it advisable to postpone the discussion of them to a relatively later time. The clean-cut division of problems into holonomic and non-holonomic, which has become possible since the work of Hertz, is very desirable and has been adopted by more than one author. The neatness of treatment of dissipative systems is much enhanced by the timely introduction of Rayleigh's dissipation function. To be sure this function does not always exist; but in the simple and probably the most important case, that in which the resistances are proportional to the velocity, it does; and gives a noteworthy simplicity to the equations of motion. The collection of thirty-five or forty problems appears to be particularly well selected and certainly presents all grades of difficulty.

The principles of Hamilton and Gauss receive careful attention in the brief space of twenty pages. The reader will perhaps find some difficulty in appreciating the methods owing to the practical impossibility of finding more than a very few problems on which to try the text. This lack may be supplied in part by having recourse to problems in earlier chapters. The author goes in some detail into the methods of the calculus of variations: He distinguishes between the applications of the principle of Hamilton to holonomic conservative systems, and its extension to dissipative and non-holonomic systems. It is in fact one of the admirable traits of this book to state with the maximum clearness just what are the conditions under which the different theorems and methods are obtained. Some treatises on dynamics merely give the equations in their simplest form and others, which presumably are more complete, fail to state the limitations under which the equations are derived. For reference purposes this is fatal. Whittaker's treatise is typographically and otherwise arranged so as to facilitate reference. To continue, however, with the present chapter we would state that the author takes up the questions of kinetic foci, of whether the integrals are actual minima (connected with the subject of fields and conjugate points in the parlance of the calculus of variations), and the way of representing dynamical systems by means of geodesics. The refinements of the modern rigorous treatment of the calculus of variations are, of course, out of the question in a work of this kind. A more natural use of space is the treatment of Bertrand's and Thomson's theorems with which the chapter closes.

The remaining one hundred and fifty pages of the book become successively more and more full of the most recent researches. Chapter X on hamiltonian systems and their integral invariants, Chapter XI on the transformation theory of dynamics with its important theorem that the whole course of a dynamical system can be regarded as the gradual self-unfolding of a contact transformation, and Chapter XII on the properties of the integrals of dynamical systems are all of tolerably recent development, although the germs of some of the methods are to be found in the works of Jacobi, Hamilton, Poisson, and even so far back as Lagrange. There are a few problems indicated for solution which must be warmly appreciated by all who know how hard it is to find specific problems in so new and advanced a domain. The last four chapters in the reduction of the problem of three bodies, the theorems of Bruns and Poincaré, the general theorem of orbits, and integration by trigonometric series are naturally essential parts of the field of mathematical astronomy, with which the most valuable part of the author's own researches have been connected. How much the student of mechanics proper may be interested in this matter is a question. The advisability of including such a treatment here cannot, however, be well open to doubt. Mathematical astronomy as one conclusion of mechanics is quite in the spirit of the present time.

Personally we should have been happy to see introduced, before these pages crammed with necessarily complicated analysis, a treatment of two problems which are of prime importance in physics and without which no student of mechanics for its applications in physics rather than in astronomy can feel himself well equipped. The first is the study of the application of Lagrange's equations and the kinetic potential to problems in physics and chemistry. This was early introduced by Maxwell into electricity and magnetism, and is treated at length in J. J. Thomson's little book on the subject. It goes a long way toward giving the lagrangian function real physical interest. Such a chapter might properly find its place just before the tenth. The second problem is that of statistical mechanics which both in theory and its applications to the kinetic theory of gases and to thermodynamics is well available, though in too great detail for the débutant, in the works of Gibbs and Jeans. A chapter on this subject, placed perhaps right after the tenth, would go far on the way toward completing a treatise on dynamics in a direction likely soon to be of greater general interest than the special methods of astronomy. May we not hope that the author's interest in astronomy (he was recently appointed astronomer royal of Ireland, a position illustrious with great names) will not prevent him from giving adequate treatment of these two physical problems in later editions of his book?

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
April, 1906.

SOME RECENT FOREIGN TEXT BOOKS.

A Course in Practical Mathematics. By F. M. SAXELBY. London, Longmans, Green, and Co., 1905. 8vo. viii + 2 unnumb. + 430 pp. Price, $2.25.

Die Planimetrie für das Gymnasium. By GUSTAV HOLZMÜLLER. Leipzig and Berlin, B. G. Teubner, 1905. 8vo. viii + 240 pp. Price, M. 2.40.

Methodisches Lehrbuch der Elementar-Mathematik. By GUSTAV HOLZMÜLLER. Leipzig and Berlin, B. G. Teubner, 1903. 8vo. xiv+370 pp. Price, M. 4.40.

Vorbereitende Einführung in die Raumlehre. By GUSTAV HOLZMÜLLER. Leipzig and Berlin, B. G. Teubner, 1904. 8vo. x+123 pp. Price, M. 1.60.

IT is probably a fact that in America we look upon the educational system of England as rather unprogressive. It is also a fact that we are accustomed to look upon its mathematics as less interesting and fruitful than that of the other leading countries of Europe. In elementary work we think of the English school as so bound to Euclid, so given to algebras of the abstract, uninteresting Todhunter type, so repelled by the practical in its adherence to what it deems the cultural, that we feel we shall find nothing suggestive in this line on British soil. And so our teachers today hear more of what is done to improve mathematics and education in France, Germany, and Italy, than in the mother country to which we are so largely indebted for our early text books and curricula. At the present time we indulge in much talk and we write numerous essays upon making our elementary mathematics more practical, upon early setting forth such of the uses of the science as are within the range of the beginner's understanding, and upon relating mathematics to the other sciences and to the general life about us, and if we give the matter a passing thought we pity England because its educators have not awakened to this vital issue.

In view of this feeling on our part, it is interesting to see what England is really doing in the way of improving elementary education in general, and mathematics in particular. Ten years have seen a very marked change of view with respect to Euclid, and although the standard classical schools still use what seem to us very uninteresting algebras and geometries, it is to England that we have to go for some of the best elementary text books we have today on the applications of mathematics. We have talked a great deal, but English teachers have acted, and neither France nor Germany is facing the issue more earnestly.

Of the several elementary text books on applied mathematics that have appeared in the last few years, Mr. Saxelby's is by far the best. It is written for institutions of a class at present more highly developed in England than here, although our technical schools are beginning to make themselves felt in a very serious way. It recognizes directly the utilitarian needs of these institutions, and in twenty-nine chapters, with less than four hundred pages, it presents the essential parts of plane trigonometry, graphs and analytic geometry, the differential and integral calculus, vector analysis, solid analytic geometry, and differential equations. This seems like a large domain, and the natural prejudgment would be that the work must be scrappy, unscientific, and incomplete. But after all, this depends upon the point of view. All elementary mathematics is scrappy, for the beginner cannot be expected to exhaust each subject before he proceeds to the next one; all is more or less unscientific, as when a pupil uses the binomial theorem before he knows anything about differentiation and convergency; and the treatment of no topic is complete. Thoroughness is a matter of degree.

Mr. Saxelby has condensed a great deal of theory into a small compass; he has eliminated what is not essential from the practical standpoint, and has arranged his material on a psychological rather than a logical plan. For example, he proves the addition formula for sin $(A+B)$ in ten lines, but only for the case of A and B acute. The general case he postpones until he takes up vector analysis. This may not seem logical, but it is very psychological, and from the standpoint of teaching it is commendable. He does little with conics, which probably brings down upon his head much criticism from the traditional schoolmaster, but he investigates a great many curves such as the engineer and the statistician need. He does not study subnormals and subtangents, but he plots $i=50 \sin 600t+20 \sin 1800\, t$ for the purpose of studying an alternating electric current, and he considers an interesting set of curves representing compound periodic oscillations, and the educational results of such work are not to be discounted by the usual remark about utilitarianism.

The fact is that a student who fairly masters a book of this kind will like mathematics better than one who puts the same amount of time on our traditional courses; he will have a far better working knowledge of the applied science; he will have cultivated more originality; he will be more desirous of studying higher mathematics; and he will not have suffered materially in the pure field. Old school mathematicians may not like the idea of making applications the basis of selection, and they may be content with that most archaic of all our sets of problems, the one generally found in our books on the calculus, but a reaction is bound to come in our college mathematics just as it has come in our school mathematics. The traditional boundaries between subjects must give way more and more, just as Mr. Saxelby has made them give way in his book; the founding of every step upon bed rock before the next step is taken must be recognized as the mission of a mature mathematician, not of a beginner who will probably not become a scientist; and the genuine applications of mathematics must replace the time-worn fictions of the 18th century writers.

This does not mean that Mr. Saxelby's work is a satisfactory text-book for American colleges. Some of his applications would not appeal to us; some of his symbolism is not ours; his text-book has not the appearance which our tastes demand; we have a different system of measures, in part, and we would probably be more traditional than he is in some respects, as in the study of certain quadratic curves. But books of this class are healthy ones for our college and preparatary teachers to have at hand; they breathe of the future; they stand for mathematics as a whole instead of its minor provinces, and for mathematics interrelated and extrarelated instead of isolated in all its parts. It is not improbable that the present agitation in the development of preparatory mathematics in this country will penetrate even into our colleges and stir up some new life. When this is done America will begin to develop for all of its schools what England is developing for its technical institutions, books of the spirit of Mr. Saxelby's, suited to the peculiar conditions and needs of our country.

In Germany of late no one has been more active in the revitalization of elementary mathematics than Professor Holzmüller. He does not represent the movement for which Mr. Saxelby stands, but his work is none the less suggestive and valuable. Germany has never been a slave to Euclid, nor have even the most conservative teachers in that country ever agreed to the interpretation of the dogma of thoroughness which has made much of English and American mathematics so repulsive. It is well known that the German schools begin their algebra and geometry much earlier than we, and extend them over a longer period, adopting a parallel instead of a tandem arrangement. It is in this elementary field, and with these principles to guide him, that Professor Holzmüller has worked.

His Planimetrie is designed for the Quarta to the Untersekunda, say the sixth to the ninth school years. As usual with German elementary text-books, it pays relatively little attention to formal definitions, but devotes much care to developing clear concepts; it is not particularly concerned with what propositions are discussed, but rather as to how they are treated; and it is not so much moved by the desire for a logical sequence as by the hope that interest, originality, and correct ideas may result from the study. As Professor Holzmüller says in his Einführung in die Raumlehre, "Erst methodische Auswahl, dann System! Erst Pestalozzi, dann Euklid!"

Brought up as we are on some revised version of Euclid, and accustomed to the traditional nomenclature, it seems strange to us to see a statement like this for a sixth-grade child: "Eine gerade Punktreihe ist eine solche, die dem Auge durch einen einzigen Punkt verdeckt werden kann." We would be apt to say either that it was too advanced for a child of eleven or twelve, or that it was not sufficiently scientific, depending upon our point of view. And yet this is a type of the statements developed for these beginners. The work is at first largely constructive, with proofs as the capabilities of the child permit. Following the advanced tendencies set forth by Méray and De Paolis there is no distinction drawn between plane and solid geometry. In the sixth school year the tetrahedron is briefly studied before the concept of angle is introduced, and most of the essential propositions of Euclid I are proved along with some treatment of elementary solids. In the Unter- and Obertertia, the seventh and eighth school years if we allow three years before the Gymnasium, plane geometry is fairly well covered, with the exception of the doctrine of similarity. Some of the work laid down would be considered too difficult for our average high schools, particularly the propositions involving cross quadrilaterals and the development of Heron's formula for the triangle. The work of Untersekunda, say the first year in our high school in point of time, relates to similar figures, the radical axis, inversion, elementary projections for map drawing, the regular solids, and the evaluation of π.

The Methodisches Lehrbuch der Elementar-Mathematik, Dritter Teil, now in its second edition, is intended for the Real and Fachschulen, and hence it involves a line of work not found in the Planimetrie. It is an excellent preparation for the study of modern mathematics, combining as it does the essentials of the recent geometry of the triangle, the theory of duality, a little projective geometry, propositions like Feuerbach's and Malfatti's on the circle and Pascal's on the hexagram in a conic, the theory of geometric involution, the theory of conics with its application to physics, stereometry, axonometry, spherics, and the application of higher algebra to geometry. Compared with Mr. Saxelby's work it involves much more mathematics and it is less practical; but on the other hand it is practical in certain directions which are not found in the English work, particularly in the theory of electricity, in higher mensuration, and in cartography. The book is one to be recommended to teachers who wish a brief general introduction to modern mathematics.

Professor Holzmüller's Vorbereitende Einführing in die Raumlehre has not so much to commend it as his other works just mentioned. It is a good book for teachers who wish to supplement the regular work in geometry by interesting side lights on the subject. Such topics as the pythagorean triangle numbers, the geometric proof concerning the sum of n odd numbers, the solids inscriptible in the regular solids, and the various types of semi-regular and stellar polyhedra are here discussed with a simplicity that renders them usable in the school room.

It should be said that none of Professor Holzmüller's works are text-books in the American sense. This is, of course, understood by all who have visited German schools or have read Professor Young's excellent work on the subject. Our type of text-book is practically unknown in the country where the pupils study and recite in the same class period and always under the master's guidance. They are, however, none the less suggestive to the American teacher, for they show what can be accomplished with pupils in certain grades, what can be omitted in the preparation of students from whom some of the world's best mathematicians are being made, what modern mathematics can replace those portions of the ancient science that are eliminated, what attention can be given to applications without interfering with the vague idea of culture which so often possesses us, and what correlation is possible among the various traditionally separate fields of the science of mathematics.

DAVID EUGENE SMITH.

NOTES.

AT the meeting of the London mathematical society held on April 26 the following papers were read: By E. B. ELLIOTT, "Perpetuants and contraperpetuants"; by A. C. DIXON, "A question in the theory of aggregates," and "The canonical forms of the ternary sextic and quaternary quartic;" by P. E. B. JOURDAIN, "On the question of the existence of transfinite numbers"; by G. H. HARDY, "Some theorems connected with Abel's theorem on the continuity of power series"; by A. R. RICHARDSON, "On a set of intervals about the rational numbers."

THE American philosophical society celebrated April 17–20 the two hundredth anniversary of the birth of its founder, Benjamin Franklin. Nearly one hundred and fifty learned societies and universities were represented at the celebration, Professor E. V. HUNTINGTON being the delegate of the AMERICAN MATHEMATICAL SOCIETY. Among the scientific papers which were presented, the following were of special interest to mathematicians: "The figure and stability of a liquid satellite," by G. H. DARWIN; "The present position of the problem concerning the first principles of scientific theory," by JOSIAH ROYCE; "Form analysis," by A. A. MICHELSON; "On positive and negative electrons," by H. A. LORENTZ.

THE next annual meeting of the Deutsche Mathematiker-Vereinigung will be held in affiliation with the seventy-eighth convention of the society of German naturalists and physicians at Stuttgart, September 16–22, under the presidency of Professor A. PRINGSHEIM, of Munich. While papers on other subjects are acceptable, principal emphasis will be put upon the theory of functions; various reports have been provided for, particularly of functions of several variables. Titles and abstracts of papers should be immediately sent to the secretary, Professor A. KRAZER, Karlsruhe in Baden, Westendstrasse 57.

THE sixth meeting of the Association of teachers of mathematics in the Middle States and Maryland was held at Columbia University on April 14. The New York section met at the same place on May 12. The Association will undertake the preparation of a bibliography of mathematical works useful to teachers. A similar undertaking was carried into effect some time ago by the Central Association. The Missouri Society of teachers of mathematics, at a meeting held at Columbia, Mo., on May 5, decided to enlarge its scope to include the sciences. Mr. J. W. WITHERS was elected president for the coming year. Provision was made for representation in the organization of a national society and for cooperation with similar societies in the discussion of the teaching of elementary physics.

THE Colorado mathematical society was organized last November, Professor I. M. DeLong being the first president. Meetings are held at Denver at intervals of about six weeks.

THE publishing house of John Wiley and Sons, New York, announce the following mathematical books in press: "Introduction to infinitesimal analysis and functions of one real variable," by O. VEBLEN and N. J. LENNES; "Textbook of mechanics," by L. A. MARTIN, volume one, Statics, volume two, Kinematics and kinetics; "Irrational numbers and their representation by sequences and series," by H. P. MANNING; "Mathematical monograms," edited by M. MERRIMAN and R. S. WOODWARD.

AMONG the books in press of the publishing house of B. G. Teubner in Leipzig, may be mentioned: Encyclopädie der mathematischen Wissenschaften, Band III, 1, Prinzipien der Geometrie, by F. ENRIQUES, Die Begriffe Linie und Fläche, by H. v. MANGOLDT, Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwickelung, by G. FANO; III, 2, Abzählende Methoden, by H. G. ZEUTHEN, Allgemeine theorie der höheren ebenen algebraischen Kurven, by L. BARZOLARI; IV, 2, Besondere Ausführungen über unstetige Bewegungen in Flüssigkeiten, by G. ZEMPLÉN, Hydraulik, by PH. FORCHHEIMER, Theorie der hydraulischen Motore und Pumpen, by M. GRÜBLER, Allgemeine Theorie der Elastizitatät, by O. TEDONE, and five articles by joint authors for volume five, parts one and two. The first three volumes of tome one of the French edition are also in press.

THE Randolph Jones company of Chicago have prepared for sale a large number of Hanstein's model and goniostat for aid in the teaching of plane and solid geometry, perspective, shadow construction, etc.

AT the University of Berlin a prize of 1000 marks has been offered for the best doctor thesis in mathematics and physics submitted to the University in 1906.

THE philosophical faculty of the University of Halle announces the following prize problem:

It is known that the determination of surfaces having constant mean curvature depends upon the same differential equation as the determination of surfaces of constant negative curvature. A number of the latter surfaces are known, and also a number of solutions of the associated partial differential equation. The surfaces of constant mean curvature which belong to these solutions are to be investigated.

Competing manuscripts should be submitted to the Secretary of the University, under the usual conditions, before October 27, 1906. The prize of 150 Marks will be awarded January 27, 1907.

THE second section of the philosophical faculty of the University of Zürich announces the following prize problem:

The motion of a material point on the surface of a sphere is to be investigated by means of at least two of the more modern methods of astronomical mechanics and the results, both analytic and numerical, compared with those obtained by means of elliptic functions.

Competing memoirs should be sent to the Rector of the University under the usual conditions by December 31, 1906.

THE following advanced courses in mathematics are announced to be given at the summer session, July 5–August 16, of the respective universities:

COLUMBIA UNIVERSITY: By Professor JAMES MACLAY: Elliptic functions.—By Professor C. J. KEYSER; Modern analytical theories in geometry.—By Professor M. W. HASKELL: Integral calculus.—By Professor EDWARD KASNER: Differential calculus.

UNIVERSITY OF PENNSYLVANIA.—By Professor G. E. FISHER: Invariants and covariants.—By Professor J. I. SCHWATT: Definite integrals.—By Professor G. H. HALLETT: Theory of abstract groups.—By Dr. F. H. SAFFORD: Differential equations.

The following advanced courses in mathematics are announced for the academic year 1906–1907:

COLUMBIA UNIVERSITY.—By Professor T. S. FISKE: Advanced calculus, introduction to the theory of functions of a real variable, three hours; Functions defined by linear differential equations, three hours. — By Professor F. N. COLE: Introduction to the theory of functions, three hours; Theory of plane curves, three hours. — By Professor D. E. SMITH: History of mathematics, two hours. — By Professor JAMES MACLAY: Application of the calculus to the theory of surfaces, three hours; Elliptic functions, three hours. — By Professor C. J. KEYSER: Modern theories in geometry, three hours; Principles of mathematics, three hours. — By Professor H. B. MITCHELL: Elements of quaternions, two hours; Differential equations, two hours. — By Professor EDWARD KASNER: Partial differential equations, two hours. — By Dr. G. H. LING: Theory of numbers, three hours; Advanced theory of numbers, three hours. — By Professor M. I. PUPIN: Theory of the potential function, two hours; Partial differential equations of physics, two hours; Hydrokinetics, two hours; Special problems, two hours. — By Professor A. P. WILLS: Mechanics, two hours; Theory of elasticity, two hours; Electricity and magnetism, electromagnetic theory of light, two hours; Thermodynamics, two hours.

Special courses in mathematical physics, of about twelve lectures each, will be given by Professor J. LARMOR, of the University of Cambridge, and Professor O. LUMMER, of the University of Breslau. Dates of these lectures and further details will be announced later.

CORNELL UNIVERSITY. — By Professor L. A. WAIT: Advanced analytic geometry, three hours; Differential calculus, II, two hours. — By Professor G. W. JONES: Algebra and imaginaries, three hours. — By Professor J. MCMAHON: Mechanics and hydrodynamics, two hours; Fourier's series and spherical harmonics, three hours. — By Professor J. H. TANNER: Theory of equations, two hours. — By Professor J. I. HUTCHINSON: Projective geometry, three hours; Seminar in automorphic functions, two hours. — By Professor V. SNYDER; Algebraic plane curves, three hours; Definite integrals, two hours. — By Professor W. B. FITE: Theory of functions of a complex variable, three hours; Theory of groups, two hours (first half year); Theory of assemblages, two hours (second half year).

HARVARD UNIVERSITY. — By Professor W. E. BYERLY:

Dynamics of a rigid body, three hours. — By Professors W. E. BYERLY and B. O. PEIRCE: Trigonometric series, introduction to spherical harmonics, the potential function, three hours. — By Professor B. O. PEIRCE: Hydromechanics, two hours. — By Professor W. F. OSGOOD: Differential and integral calculus, second course, three hours; Infinite series and products, three hours, first half-year; Algebra, Galois's theory of equations, three hours, second half-year. — By Professor M. BÔCHER: Methods of interpolation and approximation, three hours, first half-year; Theory of numbers, three hours, second half-year; Linear differential equations of physics, three hours. — By Professor C. L. BOUTON: Differential equations, with an introduction to Lie's theory of continuous groups, three hours. — By Mr. J. K. WHITTEMORE: Theory of functions (introductory course), three hours; Introduction to the differential geometry of curves and surfaces, three hours, second half-year. — By Dr. J. L. COOLIDGE: Introduction to modern geometry and modern algebra, three hours; Line geometry, three hours. — By Professor E. V. HUNTINGTON: Short course on the fundamental concepts of mathematics. — Courses in reading and research are offered by Professor BYERLY, on Some famous problems in mechanics; by Professor OSGOOD, on Topics in the theory of functions; by Professor BÔCHER, on Fluctuating functions; by Professor BOUTON, on Topics in the theory of contact transformations; by Mr. WHITTEMORE, on Goursat's Cours d'analyse; by Dr. COOLIDGE, on Enriques's Projective geometry. — The mathematical conference will meet as usual.

UNIVERSITY OF BONN. — By Professor E. STUDY: Analytic geometry; II (projective geometry), three hours; Introduction to the theory of invariants, three hours; Seminar, two hours. — By Professor G. KOWALEWSKI: Introduction to the theory of numbers, two hours; Differential and integral calculus, four hours; Critical survey of new results in the theory of assemblages, one hour; Seminar, two hours. — By Professor F. LONDON: Descriptive geometry, four hours; Theory of elliptic functions, four hours; Seminar, two hours.

THE Special Board for Mathematics have issued a report with proposals which, if carried, will have the effect of altering the character of that time-honored institution, the mathematical tripos in Cambridge (England). At present the better students take the examinations at the end of their third and fourth years, that for the latter consisting of advanced subjects only. It is proposed to divide the examinations into two parts, one of which would be taken by these men at the end of their first year: it would consist of elementary mathematics, including the calculus and the elements of dynamics, electricity and optics. The second part would be taken at the end of the third year: it is divided into two schedules. Schedule A consists of more advanced work in the same subjects and in addition, elementary theory of functions, differential equations, hydromechanics and astronomy; Schedule B contains the higher subjects. The men would therefore complete their course at the end of the third year instead of at the end of the fourth year; the additional time could be given to study without the compulsion of an examination. Students in physics and engineering would naturally take part I and schedule A of part II.

A feature of the proposals is to abolish the order of merit, arranging the names in three classes only, and attaching a special mark to the names of those who satisfy the examiners in schedule B and another special mark to those who deserve special credit in the subjects under this schedule.

Some quotations from the report may be of interest. The Board considers that under the old scheme "the ablest men . . . are encouraged to spend an excessive amount of time upon detailed work in the less advanced parts of mathematics." Again they spend "much time during the first three years of their course upon an excessive amount of polishing of their mathematical tools"; "leave themselves an insufficient amount of time for reaping the advantages of the skill they have acquired, by applying that skill to more advanced work." The educational side for others than professed mathematicians is also carefully considered.

AT the technical school of Berlin, Dr. M. REISSNER has been appointed associate professor of mechanics and Dr. G. WALLENBERG has been appointed docent in pure mathematics.

DR. C. v. WISSELINGH has been appointed professor of mathematics at the University of Groningen.

PROFESSOR J. A. GMEINER, of the German University at Prague, has been appointed professor of mathematics at the University of Innsbruck, as successor to the late Professor O. STOLZ.

PROFESSOR E. B. VAN VLECK, of Wesleyan University, has been appointed head professor of mathematics at the University of Wisconsin to succeed Professor C. A. VAN VELZER.

PROFESSOR J. A. MILLER, of the University of Indiana, has been appointed professor of mathematics and astronomy at Swarthmore College.

PROFESSOR G. A. MILLER, of Stanford University, has been appointed associate professor of mathematics at the University of Illinois.

PROFESSOR S. E. SLOCUM, of the University of Illinois, has been recalled to the University of Cincinnati, as professor of applied mathematics in the college of engineering.

PROFESSOR G. A. BLISS, of Princeton University, will become a member of the editorial board of the *Annals of Mathematics* with the beginning of the next volume in October, succeeding Professor H. S. WHITE, who retired last fall.

DR. C. N. HASKINS, of Cornell University, has been appointed assistant professor of mathematics at the University of Illinois.

AT the Massachusetts Institute of Technology the following changes are announced: Professor W. WELLS and Mr. W. S. MACDONALD have been granted leave of absence for next year; Professor F. S. WOODS has been promoted to a full professorship and Mr. N. R. GEORGE to an assistant professorship of mathematics.

PROFESSOR W. B. SMITH, of Tulane University, has been transferred from the chair of mathematics to a chair in philosophy.

DR. J. G. HARDY has been promoted to an associate professorship of mathematics at Williams College.

AT the University of Colorado Dr. SAUL EPSTEEN has been promoted to an assistant professorship of mathematics.

DR. C. E. COLPITTS, of Cornell University, has been appointed adjunct professor of mathematics at the Georgia School of Technology.

MR. L. P. SICELOF has been appointed tutor in mathematics at Columbia University.

PROFESSOR J. H. TANNER, of Cornell University, who has been absent on leave during the present academic year, will resume his work at the university in September.

MR. C. F. CRAIG has been appointed assistant in mathematics at Cornell University.

DR. O. E. GLENN has been appointed instructor in mathematics at the University of Pennsylvania.

RECENT catalogues of second-hand mathematical books: Theodor Ackermann, 10 Promenadenplatz, Munich, catalogue No. 550, 662 titles in mathematics and physics. Williams and Norgate, 14 Henrietta Street, Covent Garden, London, general science including journals, about 90 titles in mathematics. Mayér and Müller, Prinz Louis Ferdinandstrasse, 2, Berlin, N. W., catalogue No. 220, 142 pages on mathematics, physics, and astronomy; W. Heffer and Sons, Cambridge, England, catalogue No. 16, 2722 titles on mathematics, physics, and astronomy; Jacques Rosenthal, Karlstrasse, 10, Munich, catalogue No. 42, Bibliotheca Paedogogica, Part I, containing many early mathematical works. From May 21 to May 24, 1906, Southeby, Wilkinson and Hodge, of London, sold at auction the library of R. C. Fisher, Esq., which contained many Rara Mathematica, including the Treviso Arithmetic of 1478.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Alas (J. M. de las). Determinantes. Toledo, Gómez Menor, 1905. 29 pp. P. 3.00

Baffi (C.). Sulle generazioni dei complessi tetraedrali. Bologna, Cuppini, 1905. 8vo. 9 pp.

Biasi (G.). Intorno alle trasformazioni cremoniane e ad una geometria analitica di grado superiore che ne deriva. Sassari, Gallizzi, 1905. 8vo. 27 pp.

Bôcher (M.). On harmonic functions in two dimensions. 8vo. (*Proceedings of the American Academy of Arts and Sciences*, Vol. XI, pp. 577–583.)

Bowser (E. A.). Elementary treatise on the differential and integral calculus, with numerous examples. 21st edition, enlarged by 640 additional examples. New York, Van Nostrand, 1906. 12mo. 450 pp. Cloth. $2.25

Carrus (S.). Familles de surfaces à trajectoires orthogonales planes. (Thèse.) Paris, Gauthier-Villars, 1906. 4to. 93 pp.

Czuber (E.). Vorlesungen über Differential- und Integralrechnung. Band I, 2te Hälfte. 2te, sorgfältig durchgesehene Auflage. Leipzig, Teubner, 1906. 8vo. 14 + 257–560 pp. M. 6.00

Hartwig (T.). Schule der Mathematik zum Selbstunterrichte. (Neue mathematische Unterrichtsbriefe.) 3ter Band: Differential- und Integralrechnung. Wien, Perles, 1906. 8vo. 200 pp. M. 2.50

Hill (G. W.). Collected mathematical works. (In 4 volumes.) Vol. III. Washington, Carnegie Institution, 1906. 4to. 577 pp. Cloth. $2.50

Neikirk (L. I.). Groups of order p^m which contain cyclic subgroups of order p^{m-3}. (*Publications of the University of Pennsylvania*, Series in mathematics.) 8vo. 65 pp. $0.75

Pirondini (G.). Contributo alla teoria delle serie: Memoria. Parma, Rossi–Ubaldi, 1905. 8vo. 32 pp.

Pompeiu (D.). Sur la continuité des fonctions de variables complexes. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 53 pp. Fr. 3.00

Rilly (A.). Le problème du cavalier des échecs. Etude sur la symétrie latérale en deux chaînes fermées. Paris, Rilly, 1906. 8vo. 108 pp.

Sachs (J.). Projektivische (neuere) Geometrie. Nach System Kleyer bearbeitet. (3 Teile.) Teil III: Pol und Polare; Mittelpunktseigenschaften; Involution; Brennpunktseigenschaften der Kurven 2ten Grades. (Etwa 25 Hefte.) Heft I. Bremerhaven, 1906. 8vo. M. 0.25

Simon (M.). Analytische Geometrie der Ebene. (Sammlung Göschen, 65.) 2te, verbesserte Auflage. Leipzig, Göschen, 1906. 8vo. 196 pp. Cloth. M. 0.80

Tannery (J.). Leçons d'algèbre et d'analyse, à l'usage des élèves des classes de mathématiques spéciales. Tome 2. Paris, Gauthier-Villars, 1906. 8vo. 640 pp. Fr. 12.00

Zoretti (L.). Sur les fonctions analytiques uniformes qui possèdent un ensemble parfait discontinu de points singuliers. (Thèse.) Paris, Gauthier-Villars, 1905. 4to. 53 pp.

II. ELEMENTARY MATHEMATICS.

Arzelà (C.). Trattato di algebra elementare ad uso dei licei e degli istituti tecnici. 3a edizione, completamente rifatta. Firenze, 1906. 16mo. 15 + 503 pp. L. 3.50

Bourlet (C.) et Hulot (J.). Précis d'algèbre, rédigé conformément aux programmes du 27 juillet 1905. Paris, Hachette, 1906. 16mo. 507 pp. Fr. 3.50

Buck (R. C.). A manual of trigonometry. 3rd edition, revised and corrected. London, Griffin, 1906. 8vo. 124 pp. 3s. 6d.

Cracknell (A. G.). See Workman (W. P.).

Dicknether (F.). Lehrbuch der Arithmetik nebst Uebungsaufgaben für Mittelschulen. 2ter Teil. 2te Auflage. München, Lindauer, 1906. 8vo. 4 + 125 pp. M. 1.80

Faifofer (A.). Elementi di algebra ad uso della prima classe liceale. Venezia, Sorteni, 1905. 16mo. 193 pp. L. 2.00

Gazzaniga (P.). See Veronese (G.).

Haas (A.). Der binomische und polynomische Lehrsatz, die arithmetischen Reihen höherer Ordnung und die unendlichen Reihen. Nach System Kleyer bearbeitet. Teil I: Der binomische Lehrsatz für ganze positive Exponenten. (Etwa 27 Hefte.) Heft 1. Bremerhaven, 1905. 8vo. M. 0.25

Holtze (A.). See Köstler (H.).

Hulot (J.). See Bourlet (C.).

Kohlschütter (E.). Messkarte zur Auflösung sphärischer Dreiecke. Nach Chauvenet neu herausgegeben. Berlin, 1906. M. 1.50

Köstler (H.). Leitfaden der ebenen Geometrie für höhere Lehranstalten, neu herausgegeben von A. Holtze. 1tes Heft: Kongruenz. 6te, teilweise umgearbeitete Auflage. Halle, Nebert, 1906. 8vo. 86 pp. M. 1.35

Lalande (J. de). Tables des logarithmes. Avec additions et modifications par E. E. Neël. 10e édition. Louvain, 1905. 8vo. 210 pp. Fr. 3.50

Legendre (A. M.). Compendio di geometria: libro di testo per le scuole tecniche conforme ai programmi governativi, ricavato dagli Elementi di A. M. Legendre per cura di G. Tolomei. Firenze, Ricci, 1905. 8vo. 179 pp. L. 1.75

Lemaire (G.). Méthodes de résolution et de discussion des problèmes de géométrie. 2e édition. Paris, Vuibert, 1906. 8vo. 227 pp.

Lombardo (S. G.). Elementi di geometria per gli aspiranti alla reale scuola militare di Modena. Messina, Davì, 1906. 16mo. 101 pp. L. 1.50

Lübsen (H. B.). Ausführliches Lehrbuch der Arithmetik und Algebra zum Selbstunterricht und mit Rücksicht auf die Zwecke des praktischen Lebens bearbeitet. 26te Auflage. Leipzig, Brandstetter, 1906. 8vo. 6 + 261 pp. M. 4.50

Neél (E. E.). See Lalande (J. de).

Purser (F.). Elementary geometry based on Euclid's Elements. London, Longmans, 1906. 12mo. 2s. 6d.

Sasso (M.) Formole della quarta e quinta potenza dei polinomi e loro applicazione. Avellino, Pergola, 1905. 8vo. 21 pp.

Simon (M.). Methodik der elementaren Arithmetik in Verbindung mit algebraischer Analysis. Leipzig, Teubner, 1906. 8vo. 108 pp. M. 3.20

Simón y Mayorga (J.). Nociones de geometría, seguidas de una colección de ejercicios y problemas. Salamanca, Iglesias, 1906. 114 pp. P. 2.50

Socci (A.) e Tolomei (G.). Nozioni intuitive di geometria elementare: libro di testo per il ginnasio inferiore secondo i vigenti programmi. 2a impressione. Firenze, Fiorentina, 1906. 8vo. 112 pp. L. 1.00

Tolomei (G.). See Legendre (A. M.), and Socci (A.).

Vagnier et Vernadet. Algèbre. Paris, Delagrave, 1906. 8vo. 275 pp.

Vernadet. See Vagnier.

Veronese (G.). Elementi di geometria ad uso dei ginnasi e licei e istituti tecnici, trattati con la collaborazione di P. Gazzaniga. Parte II. 3a edizione. Padova, Prosperini, 1905. 8vo. 220 pp. L. 2.25

Wells (W.). Algebra for secondary schools. Boston, Heath, 1906. 12mo. 10 + 462 pp. Half leather. $1.20

Workman (W. P.) and Cracknell (A. G.). Preparatory course in geometry. 2nd edition. London, Clive, 1906. 8vo. 76 pp. 9d.

III. APPLIED MATHEMATICS.

Aalst (J. W. van). Theoretische beschouwingen over het warmteproces in stoomturbines, petroleum- en gasmotoren en verkoelingsmachines, met korte beschrijving van enkele soorten dier machines. Helder, 1905. 8vo. 4 + 119 pp. M. 3.00

Ancey (C.). Théorie et pratique des opérations d'assurance. 2e édition, entièrement refondue et complétée. Paris, 1906. 8vo. 437 pp. Fr. 7.00

Antomari (X.). Cours de géométrie descriptive. 3e édition, conforme au nouveau programme de mathématiques spéciales. Paris, Vuibert, 1906. 8vo. 623 pp.

Biscan (W.). Die Starkstromtechnik. (2 Bände.) Band I: Gesetze und Erzeugung der elektrischen Energie. Leipzig, 1906. 8vo. 10 + 488 pp. M. 15.00

Broggi (U.). Matematica attuariale. Storia statistica della mortalità. Matematica delle assicurazioni sulla vita. Milano, 1906. 8vo. 15 + 347 pp. L. 3.50

Cappa (S.). Corso di meccanica applicata alle macchine. Torino, 1905. 8vo. 658 pp. L. 32.50

Clouth (F. M.). Tables pour le calcul des coordonnés goniométriques. 3e édition, revue et corrigée. Halle, Nebert, 1906. 8vo. 8 + 201 pp. M. 7.00

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NOTE ON THE NUMERICAL TRANSCENDENTS S_n AND $s_n = S_n - 1$.

BY PROFESSOR W. WOOLSEY JOHNSON.

1. The numbers defined by the series

$$S_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots,$$

where n is a positive integer greater than unity, are of frequent occurrence in analysis. Euler, in the Institutiones Calculi Differentialis, 1775, gave a table of their values up to S_{16} to sixteen places of decimals (page 456). He had connected the values of the even-numbered ones with Bernoulli's numbers and the even powers of π by the formula

$$S_{2n} = \frac{2^{2n-1} B_n \pi^{2n}}{(2n)!},$$

but failed to obtain a finite expression for the odd-numbered ones. He gave also the value of the constant γ, now known as Euler's constant, defined as the limiting value

$$\gamma = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \log n \right]_{n=\infty}.$$

2. The constants γ and S_n occur in the expressions for the function $\log \Gamma(1+x)$; and Legendre, for the purpose of constructing his table of $\log \Gamma(a)$, examined Euler's table of values for S_n and, finding "quelques erreurs assez graves," reconstructed the table, carrying it to S_{35}, remarking that for higher values of n one has only to divide the excess over unity successively by 2. This is of course because the powers of 1/3, 1/4, etc., have disappeared from the last retained decimal place, which was as in Euler's table the sixteenth.

The values of S_n rapidly approach that of the first term, which is unity. It follows that, an algebraic series being given in which the S_n's occur, if the series resulting from replacing S_n by unity has been already summed, we can by subtraction obtain a much more rapidly convergent series. Thus Legendre replaced his series

$$\log\Gamma(1+x)=\tfrac{1}{2}\log\frac{\pi x}{\sin\pi x}-\gamma x-\tfrac{1}{3}S_3x^3-\tfrac{1}{5}S_5x^5-\cdots$$

by

$$\log\Gamma(1+x)=\tfrac{1}{2}\log\frac{\pi x}{\sin\pi x}-\tfrac{1}{2}\log\frac{1+x}{1-x}$$
$$+(1-\gamma)x-\tfrac{1}{3}(S_3-1)x^3-\tfrac{1}{5}(S_5-1)x^5-\cdots$$

3. I propose in this note to put $S_n=1+s_n$, so that

$$s_n=\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\cdots.$$

A table of values of S_n is of course virtually a table of values of s_n.

When values of x for which the Γ-function is known are substituted in the algebraic series, numerical series involving S_n and γ are found; thus Legendre* derives from the above equation by putting $x=1$ and $x=\tfrac{1}{2}$,

$$(1)\qquad 1-\gamma=\tfrac{1}{2}\log 2+\tfrac{1}{3}s_3+\tfrac{1}{5}s_5+\tfrac{1}{7}s_7+\cdots,$$

$$(2)\qquad 1-\gamma=\log\tfrac{3}{2}+\frac{s_3}{3\cdot2^2}+\frac{s_5}{5\cdot2^4}+\frac{s_7}{7\cdot2^6}+\cdots.$$

Legendre computed γ from each of these series by means of his table of S_n, and cites the agreement with the known value of γ as a test of the accuracy of the table.

Dr. Glaisher in a paper "On the history of Euler's constant," *Messenger of Mathematics*, 1871, cites a number of these relations from the Memoirs of Euler, one of which is, in the present notation,

$$(3)\qquad \gamma=\tfrac{1}{2}s_2+\tfrac{2}{3}s_3+\tfrac{3}{4}s_4+\cdots.$$

This formula was given in 1769, and in a memoir of 1781 occurs the formula

$$(4)\qquad 1-\gamma=\tfrac{1}{2}s_2+\tfrac{1}{3}s_3+\tfrac{1}{4}s_4+\cdots.$$

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*Traité des fonctions elliptiques et des intégrales eulériennes, vol. 2, p. 434. The table of values of S_n is on p. 432.

4. It was a comparison of these equations which first led me to notice the very simple relation, independent of γ,

$$1 = s_2 + s_3 + s_4 + \cdots. \tag{5}$$

This relation must doubtless have been frequently noticed by those who have had occasion to deal with these numbers,* yet it seems clear that it was not known to Legendre writing in 1826, for it forms a much better verification of his table (involving, as it does, every figure of it) than does the computation of γ mentioned above.

5. It is the main object of this note to point out that not only this result but the other results mentioned above involving γ and naperian logarithms may be derived by direct summation of series in s_n's.

Thus, if we write out the terms of each s_n in a column, we have

$$\begin{aligned}\sum_2^\infty s_n = & (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 + \cdots \\ & + (\tfrac{1}{3})^2 + (\tfrac{1}{3})^3 + (\tfrac{1}{3})^4 + \cdots \\ & + (\tfrac{1}{4})^2 + (\tfrac{1}{4})^3 + \cdots \\ & + \cdots.\end{aligned}$$

The rows form geometric progressions, hence we have

$$\begin{aligned}\sum_2^\infty s_n &= \frac{(\frac{1}{2})^2}{1-\frac{1}{2}} + \frac{(\frac{1}{3})^2}{1-\frac{1}{3}} + \frac{(\frac{1}{4})^2}{1-\frac{1}{4}} + \cdots \\ &= \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \cdots = 1.\end{aligned}$$

In like manner we can show that

$$s_2 - s_3 + s_4 - \cdots = \tfrac{1}{2};$$

whence it follows that

$$s_2 + s_4 + s_6 + \cdots = \tfrac{3}{4}, \tag{6}$$

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* I have, since writing the above, found it given as a problem in the second edition of Boole's Finite Differences in the form : "Shew that the sum of all the negative powers of all whole numbers (unity being in both cases excluded) is unity ; if odd powers are excluded, it is $\frac{3}{4}$."

and

$$s_3 + s_5 + s_7 + \cdots = \tfrac{1}{4}. \tag{7}$$

Legendre's table is completely verified by these equations, the sums being in fact only 2 or 3 units out in the 16th place.

Again, to sum $\frac{1}{2}s_2 + \frac{1}{3}s_3 + \cdots$, we have

$$\begin{aligned}\sum_2^\infty \frac{s_n}{n} = {} & \tfrac{1}{2}(\tfrac{1}{2})^2 + \tfrac{1}{3}(\tfrac{1}{2})^3 + \tfrac{1}{4}(\tfrac{1}{2})^4 + \cdots \\ & + \tfrac{1}{2}(\tfrac{1}{3})^2 + \tfrac{1}{3}(\tfrac{1}{3})^3 + \cdots \\ & + \tfrac{1}{2}(\tfrac{1}{4})^2 + \tfrac{1}{3}(\tfrac{1}{4})^3 + \cdots \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

Here each row is of the form

$$\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \log \frac{1}{1-x} - x,$$

where x has the values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., in the successive rows. Hence

$$\begin{aligned}\sum_2^\infty \frac{s_n}{n} &= \Big[\log 2 + \log \tfrac{3}{2} + \log \tfrac{4}{3} + \cdots + \log \frac{m}{m-1} \\ &\qquad - \Big(\tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{m}\Big)\Big]_{m=\infty} \\ &= \Big[\log m - \Big(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{m}\Big)\Big]_{m=\infty} + 1 = 1 - \gamma,\end{aligned}$$

which is equation (4) above.

6. Treating the sum $s_2 + \frac{1}{2}s_4 + \frac{1}{3}s_6 + \cdots$ in the same manner, the rows are of the form

$$x^2 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^6 + \cdots = \log \frac{1}{1-x^2},$$

x having the successive values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. Thus

$$\begin{aligned}\sum_1^\infty \frac{s_{2n}}{n} &= \log \tfrac{4}{3} + \log \tfrac{9}{8} + \log \tfrac{16}{15} + \cdots \\ &= \log \left[\tfrac{2}{1}\ \tfrac{2}{3}\ \tfrac{3}{2}\ \tfrac{3}{4}\ \tfrac{4}{3}\ \tfrac{4}{5} \cdots\right] = \log 2.\end{aligned} \tag{8}$$

As a further example, Legendre's equation, (2) of this note, may be verified. Thus in summing $\frac{s_3}{3.2^2} + \frac{s_5}{5.2^4} + \cdots$, the rows take the form, if $y = \frac{1}{2}x$,

$$\frac{x^3}{3 \cdot 2^2} + \frac{x^5}{5 \cdot 2^4} + \frac{x^7}{7 \cdot 2^6} = 2\left[\frac{y^3}{3} + \frac{y^5}{5} + \cdots\right] = \log\frac{1+y}{1-y} - 2y$$

where y takes the successive values $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, etc. Hence

$$\begin{aligned}\sum_1^\infty \frac{s_{2n+1}}{(2n+1)2^{2n}} &= \log\left[\tfrac{5}{3}\ \tfrac{7}{5}\ \tfrac{9}{7}\cdots\tfrac{2n+1}{2n-1}\right]_{n=\infty} \\ &\qquad - \left[\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{n}\right]_{n=\infty} \\ &= \left[\log(2n+1) - \log 3 + 1\right. \\ &\qquad \left. - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n}\right)\right]_{n=\infty}\end{aligned}$$

and, since $\log(2n+1)$ at the limit becomes $\log 2 + \log n$, the result is

$$\sum_1^\infty \frac{s_{2n+1}}{(2n+1)2^{2n}} = \log\tfrac{2}{3} + 1 - \gamma,$$

which is equation (2).

7. I have been tempted to make an independent calculation of the values of s_n, using 12 decimal places. The values of s_n when $n > 9$, when carried thus far, are readily obtained by direct summation of all their terms. But the extreme slowness of the convergence of the terms when n is small renders it practically necessary to employ the same method as that by which γ has been computed, namely the Euler-Maclaurin formula for the summation of a finite number of terms, viz.:

$$\Sigma u_x = C + \int u_x dx - \tfrac{1}{2}u_x + \frac{B_1}{2}\frac{du_x}{dx} - \frac{B_2}{4!}\frac{d^3u_x}{dx^3} + \frac{B_3}{6!}\frac{d^5u_x}{du^5} - \cdots$$

In the application of this formula when u_x is a negative power of x, the constant C is the sum to infinity, and is obtained by direct summation of Σu_x to a moderate value of x, and calculation of the infinite series in the second member. It was thus that Euler and Legendre calculated the value of γ as well as the values of S_n, and thus that Adams, after greatly extending the range of known Bernoullian numbers, calculated γ to 263 place of decimals.

Inasmuch as Legendre's table has not often been reprinted, it may be of interest to give the results of my computation to the eleventh place of decimals. They are as follows:

$$\text{Values of } s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$$

n	s_n	n	s_n
2	.64493 40668 5	14	.00006 12481 4
3	.20205 69031 6	15	3 05882 4
4	8232 32337 1	16	1 52822 6
5	3692 77551 4	17	76372 0
6	1734 30619 8	18	38172 9
7	834 92773 8	19	19082 1
8	407 73562 0	20	9539 6
9	200 83928 3	21	4769 3
10	99 45751 3	22	2384 5
11	49 41886 0	23	1192 2
12	24 60865 5	24	596 1
13	12 27133 5	25	298 0

[The values for $n > 25$ are obtained each by dividing its predecessor by 2.]

ANNAPOLIS,
May, 1906.

ON CERTAIN PROPERTIES OF WRONSKIANS AND RELATED MATRICES.

BY PROFESSOR D. R. CURTISS.

(Read before the Chicago Section of the American Mathematical Society, April 14, 1906.)

In this note I shall present theorems of a very general character on the vanishing of Wronskians and related matrices. Proofs, however, will be reserved for subsequent publication in more extended form.

Let $u_1, u_2, \cdots, u_n$ be functions, real or complex, of the real variable x, having finite derivatives of the first k orders

$(k \geqq n-1)$ at each point of an interval I. This interval may be finite or infinite, and, if limited in either direction, may or may not include its end points. We shall use the symbol $M_k(u_1, u_2, \cdots, u_n)$ to designate the matrix

$$\left|\begin{matrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ u_1^{(k)} & u_2^{(k)} & \cdots & u_n^{(k)} \end{matrix}\right|$$

The Wronskian $W(u_1, u_2, \cdots, u_n)$ will then be the n-rowed determinant whose matrix is $M_{n-1}(u_1, u_2, \cdots, u_n)$.

The first three of the following theorems concern the vanishing of the Wronskian in an infinite set of points belonging to the interval I, and having a limiting point p which is also a point of I. The symbol $[P]$ will be used to designate such a set when it includes the point p. A set which does not contain the limiting point p will be referred to as a set $[P^*]$.

THEOREM I. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have finite derivatives of the first* n *orders; then if* $W(u_1, u_2, \cdots, u_n)$ *vanishes in a point set* $[P]$ *at least one of the Wronskians* $W(u_1', u_2', \cdots, u_\nu')$ $(\nu = 1, 2, \cdots, n)$ *vanishes in a set* $[P^*]$.

THEOREM II. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have continuous derivatives of the first* k *orders* $(k \geqq n)$*; then if* $W(u_1, u_2, \cdots, u_n)$ *vanishes in a point set* $[P]$*, all the* n*-rowed determinants of the matrix* $M_k(u_1, u_2, \cdots, u_n)$ *vanish at the point* p.

In this last theorem the functions $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are supposed continuous. The following theorem does not make this hypothesis:

THEOREM III. *Let* $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)} (k \geqq n)$ *exist and be finite at every point of* I*; then if* $W(u_1, u_2 \cdots, u_n)$ *and its first* $k-n+1$ *derivatives vanish simultaneously in a point set* $[P]$ *all the* n*-rowed determinants of the matrix* $M_k(u_1, u_2, \cdots, u_n)$ *vanish simultaneously either at* p *or in a set* $[P^*]$.

From these three theorems we deduce the following, which is easily seen to be a direct consequence of Theorem II when $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are continuous:

THEOREM IV. *Let* $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)} (k \geqq n)$ *exist and be finite at every point of* I*; then if* $W(u_1, u_2 \cdots, u_n)$ *vanishes identically throughout* I *every* n*-rowed determinant of* $M_k(u_1, u_2, \cdots, u_n)$ *vanishes identically throughout* I.

For an interval where $W(u_1, u_2, \cdots, u_{n-1})$ does not vanish, Theorem IV has been stated and proved by Bôcher*; this restriction is not necessary to my proof. As a corollary we obtain Bôcher's theorem that if $W(u_1, u_2, \cdots, u_n) \equiv 0$ then $W(u_1, u_2, \cdots, u_{n+1}) \equiv 0$, but Bôcher's assumption of the continuity of $u_1^{(n)}, u_2^{(n)}, \cdots, u_{n+1}^{(n)}$ is now clearly seen to be unnecessary.

The preceding theorems have an important application in the theory of linear dependence. The results which I have obtained are summarized in the following theorem where the phrase, "$M_k(u_1, u_2, \cdots, u_n)$ is of constant rank $m < n$ in I," means that every $(m + 1)$-rowed determinant of this matrix vanishes identically throughout I while the m-rowed determinants do not all vanish at any point of I:

THEOREM V. *If* $M_k(u_1, u_2, \cdots, u_n)$ *is of constant rank* $m < n$ *in* I, $u_1, u_2, \cdots, u_n$ *are linearly dependent and the number of independent linear relations between these functions is* $n - m$.

Since $M_k(u_1, u_2, \cdots, u_n)$ is of constant rank if $u_1, u_2, \cdots, u_n$ are solutions of a linear differential equation of order k, it is evident that Theorem V is of wide application. In fact it includes all sufficient conditions for the linear dependence of non-analytic functions that have been given by Peano and Bôcher, but applies to no cases which do not come under Bôcher's Theorem VI.† This theorem states that if $W(u_1, u_2, \cdots, u_n) \equiv 0$, while no function (other than zero) of the form

$$g_1 u_1 + g_2 u_2 + \cdots + g_n u_n$$

(the g's being constants) vanishes together with its first k derivatives at any point of I, the functions $u_1, u_2, \ldots, u_n$ are linearly dependent. With the aid of Theorem V it can be shown that a necessary and sufficient condition for the verification of the hypothesis of Bôcher's theorem is that $M_k(u_1, u_2, \cdots, u_n)$ be of constant rank $m < n$.

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* *Transactions Amer. Math. Society*, vol. 2 (1901), p. 139. In this article the author gives references to other papers on Wronskians by Peano and himself.

† *Loc. cit.*, p. 144.

NORTHWESTERN UNIVERSITY,
May, 1906.

SIGNIFICANCE OF THE TERM HYPERCOMPLEX NUMBER.

BY PROFESSOR JAMES BYRNIE SHAW.

(Read before the Chicago Section of the American Mathematical Society, April 14, 1906.)

ESSENTIALLY four definitions of quite different logical import have been given for the term hypercomplex number, or multiple number. The four things so defined differ considerably in their mathematico-philosophical meaning, and while two of them are in a way equivalent, neither of the others can be correlated with these two or with each other as equivalent. It is proposed to examine these four definitions rather critically.

I. *The n-tuple.*

The first definition I shall denominate the Dedekind definition, although Hamilton discussed, many years before, entities defined in the same way. It is of a pure arithmetical character, since it implies only the existence of a set of things we may call numbers, marks, or entities, according as we conceive them to belong to a domain of integrity, an abstract field, or, in general, an aggregate that we can call a range. At first these entities were in a scalar domain, then they were generalized to a rational domain, then to an abstract field, and obviously we may take them from any range. The definition runs substantially thus : [*]

A set of n ordered marks (entities) $a_1, \cdots, a_n$ of a field (range) F, is called an n-tuple element a. The symbol $a = (a_1, \cdots, a_n)$ employed is purely positional without functional connotation. Its definition implies that $a = b$ if, and only if, $a_1 = b_1, \cdots, a_n = b_n$.

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[*] Dickson, "On hypercomplex number systems" ; *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 344–348.

This system of n-tuple elements is then made to be linear and associative by a set of postulates of such a nature that the sum and the product of any two elements is a third element. Like all arithmetic definitions, this definition has the merit of being free from existential questions. If marks of a field (or entities of a range) exist, these elements exist. The element is a conception which is a function of its n marks, or n coördinates. The conception is one needed in many applications of mathematics, hence its introduction is justified. But logically these elements are not extensions of the ordinary numbers and can be so called only by an extension of the meaning of the term number, and this extension must be in an altogether artificial manner. As Bertrand Russell points out with regard to the elements defined in the next definition, we cannot identify * unity with any of the n-tuple elements, not even with the modulus, which may be written

$$(1, 0, 0, \cdots, 0).$$

For unity is not n-tuple, not even an n-tuple with $n - 1$ zero coördinates. The distinction is analogous to that which insists that unity and positive unity are two different conceptions.

The n-tuple has no reference to anything else. It is a compound conception built up directly from the range given. Its laws of combination are purely arbitrary, since there is no reason à priori why, in $(a, b)(x, y) \equiv (\gamma_{111}ax + \gamma_{121}ay + \gamma_{211}bx + \gamma_{221}by, \gamma_{112}ax + \gamma_{122}ay + \gamma_{312}bx + \gamma_{222}by)$, the eight γ's should have one value rather than another. The thing to do, on this basis, obviously is to study all possible types of algebras of such n-tuples, and find their uses, if any.

II. *The Manifold.*

The second definition I shall call the Russell definition, although it has appeared before in works of other logicians. It runs thus:

What defines a hypercomplex number is a one-many relation whose domain consists of real numbers (that is, ordinary rationals, irrationals, etc.), and whose converse domain consists of the first n integers (or, in the case of complex numbers of infinite order, of all integers). These relations are indicated by $e_1, e_2, \cdots, e_n$. Thus, in $a = a_1e_1 + a_2e_2 + \cdots + a_ne_n$, a_1 is correlated to 1, a_2 to 2, $\cdots, a_n$ to n. The relation is one-many because a_i may equal a given number b, for $i = 1, 2, \cdots, n$; that is, b is correlated to each integer; so for every number. These one-many relations may be defined to be the complex numbers. In this way a purely logical definition is obtained. We are not to identify $1\,e_1 + 0e_2 + \cdots + 0e_n$ with e_1, that is, the complex number is a function of the coordinates, not of the relations e. Evidently we may use any other range than that of all real numbers, and this definition becomes as general as the first, which it most resembles. Russell proceeds immediately to show that this virtually defines a complex number to be an n-dimensional entity. It is substantially the definition at the foundation of vector analysis, where the relational symbols $e_1, e_2, \cdots, e_n$ are expressed by directions.

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* Principles of Mathematics, vol. 1, p. 380.

It is sufficiently clear that these manifolds are not in the ordinary number system any more than were the n-tuples.

Their sums and their products are likewise manifolds whose coordinates are determined by certain laws of combination. True, we may call them numbers, but the identification of an ordinary number with any one of the manifolds is highly artificial. In fact, students of space analysis or vector analysis would scarcely conceive of any such identification being possible, any more than they would think of identifying an angle with its number of degrees.

The connection of the n-tuple and the manifold with the bilinear function of n variables is evident.

This is the point of view of the qualitative-unit mathematician, a unit 1 correlated to the n integers gives the n units $1e_1, 1e_2, \cdots, 1e_n$, of the system. These are usually written $e_1, \cdots, e_n$, although $1e_1$ is not really e_1; for e_1 really expresses only that something is correlated to 1. The correlation may as well be with n directions in space, and might be written $i, j, k, \cdots$. In this case the qualitative phase is a little more apparent. Of course we are left free to say what $ae_1 \times be_2$ shall mean; this gives rise to a multiplication table, meaning that if a is correlated to 1, b to 2, then ab shall be correlated to $\gamma_{121}e_1 + \gamma_{122}e_2 + \cdots + \gamma_{12n}e_n$. This view appeals strongly to the quaternionist or vector analyst who desires a flexible symbolism, although many use the cumbrous n-tuple forms.

III. *The Operator.*

The third definition I shall call the Peano definition, although it appeared long before in Hamilton's quaternions. This definition looks upon the hypercomplex number as an operator. What Peano* calls a complex number is incapable of multiplication, as he explicitly says, hence it is not the kind of entity we are discussing here. But he does not include quaternions, $\sqrt{-1}$, and the like, in the class of complex numbers. These he defines as operators. Thus, $\sqrt{-1}$ is the substitution $\begin{vmatrix} 0 & -1 \\ +1 & 0 \end{vmatrix}$, that is, if we have given a non-multiplicative manifold (x, y), then by $\sqrt{-1}\ (x, y)$ we mean $(-y, x)$. Obviously matrices come under this head, as do linear substitutions.

It will readily be granted, I suppose, that these operators are not numbers, although Hamilton identifies quaternions that are purely scalars with the domain of positive and negative numbers. But he means that the scalar 1 is an operator such that applied to a, any vector, $1\,a = a$. If we were to define numbers as operators, then, of course, scalars are numbers. But numbers defined on a pure arithmetical basis are not operators.

It is to be noted carefully in these three definitions that the product or sum of any two elements is a third element of the same character. No possible combination ever leads us away from the range under consideration. In the first case we have, for example, the products of two n–tuples: $(a, b)(x, y) = (ax\gamma_{111} + ay\gamma_{112} + bx\gamma_{121} + by\gamma_{122}, ax\gamma_{211} + ay\gamma_{212} + bx\gamma_{221} + by\gamma_{222})$, an n–tuple. To connect these with numbers, we virtually must define real numbers to be n-tuples. In the second case we have for the product of two manifolds $(ae_1 + be_2)(xe_1 + ye_2) = [(ax\gamma_{111} + ay\gamma_{112} + bx\gamma_{121} + by\gamma_{122})\,e_1 + (ax\gamma_{211} + ay\gamma_{212} + bx\gamma_{221} + by\gamma_{222})e_2]$, a manifold. To connect with numbers, we must define numbers as manifolds. In the third case we have for the product of two operators $(ai_1 + bi_2)(xi_1 + yi_2)(\ ) = [ax\gamma_{111} + ay\gamma_{113} + bx\gamma_{121} + by\gamma_{122})\,i_1 + (ax\gamma_{211} + ay\gamma_{212} + bx\gamma_{221} + by\gamma_{222})\,i_2](\ )$, an operator. To connect with numbers, we must define a number as an operator.

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*[*Formulaire* (1901).

IV. *The Congruence.*

The fourth definition I shall call the *algebraic* definition. It is so called by Russell, who says indeed regarding this method* ". . . the algebraic generalisation is very inferior to the arithmetical, since the latter gives all irrationals by a uniform method, whereas the former, strictly speaking, will give only the algebraic numbers. But with regard to the complex numbers, the matter is otherwise. No arithmetical problem leads to these, and they are wholly incapable of arithmetical definition. But the attempt to solve such equations as $x^2 + 1 = 0$, $x^2 + x + 1 = 0$ at once demands a new class of numbers, since, in the whole domain of real numbers, none can be found to satisfy these equations. To meet such cases, the algebraic generalisation defined new numbers by means of the equations whose roots they were. . . . But the algebraic generalisation was wholly unable (as it was, in truth, at every previous stage) to prove that there are such entities as those which it postulated."

Without agreeing or disagreeing with this statement as it stands, we may place the algebraic definition on the following unimpeachable basis: † Let us consider that i is such an entity that wherever we find the expression $i^2 + 1$ we may write 0 for it; that is, we reduce all our problems, as it were, modulo $i^2 + 1$. Then we may write $x^2 + x + 1 = x^2 + x + \frac{1}{4} + \frac{3}{4} = x^2 + x + \frac{1}{4} + \frac{3}{4} - \frac{3}{4}(i^2 + 1) = x^2 + x + \frac{1}{4} - \frac{3}{4}i^2$. This evidently has the factors $x + \frac{1}{2} - \frac{1}{2}\sqrt{3}\,i$ and $x + \frac{1}{2} + \frac{1}{2}\sqrt{3}\,i$. Hence we may speak of the roots as $-\frac{1}{2} + \frac{1}{2}\sqrt{3}\,i$ and $-\frac{1}{2} - \frac{1}{2}\sqrt{3}\,i$. We have not made any assertions at all regarding the nature of i, and may if we choose look upon it as simply a perfectly arbitrary number, or algebraic *form* destitute of numerical value. We may look upon all radicals in the same way; thus $x^2 + 6x + 7$ has no rational roots. But if we work modulo $j^2 - 2$, we find at once that we may write $x^2 + 6x + 7 = x^2 + 6x + 9 - j^2$, and we say $x + 3 \pm j = 0$. We call j an extension of our number system. So we call i an extension of the number system.

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* Principles of Mathematics, p. 377 et seq.

† With regard to this basis, I desire to acknowledge valuable suggestion from Professors Moore and Dickson.

If now we go farther and work modulo $i^2+1, j^2+1, k^2+1, l^2+1, \cdots$, where $i, j, k, l, \cdots$ are different forms, we find that we may take $(i+j)^2+2=0$ if $ij+ji=0$. Hence it is suggested that we must either consider $ij=0$, or else consider that ij and ji are not equal. That is, we must choose between nil-factors with commutativity, or else give up commutativity. We do the latter and arrive at quaternions. So that we have expressions reducible modulo

$$\begin{gathered} i_1^2+1, \quad j_1^2+1, \quad k_1^2+1, \quad i_2^2+1, \quad j_2^2+1, \quad k_2^2+1, \cdots, \\ i_1j_1+j_1i_1, \quad j_1k_1+k_1j_1, \quad k_1i_1+i_1k_1, \quad i_2j_2+j_2i_2, \cdots, \\ i_1j_2, \quad i_2j_1, \quad i_1k_2, \quad i_2k_1, \quad j_1k_2, \quad j_2k_1, \quad j_1i_2, \quad j_2i_1, \\ k_1i_2, \quad k_2i_1, \quad k_1j_2, \quad k_2j_1, \quad \cdots. \end{gathered}$$

If we use only i_1, j_1, k_1, we find we may factor the expression

$$x^2-2ax+a^2+b^2+c^2+d^2,$$

namely

$$x^2-2ax+a^2+b^2+c^2+d^2=(x-a+bi_1+cj_1+dk_1)(x-a-bi_1-cj_1-dk_1).$$

We have thus arrived at quaternions. Here again we may consider i_1, j_1, k_1 as merely formal, or we may look on them as extensions of the number system. In the end the latter view seems more profitable.

We may then state the algebraic definition for the general case as follows: Let there be an expression, or set of expressions, in terms of $q, r, \cdots$ such that we reduce all expressions in terms of the entities of a given range, modulo these given expressions, where $q, r, \cdots$ represent any entities for which this can be done. Then the entities q, say $q_1, q_2, \cdots$, and the entities r, say $r_1, r_2, \cdots$, etc., define extensions of the given range. If the number of these ideal entities is finite we arrive at an algebra of a finite number of dimensions, otherwise an algebra of infinite dimensions.

V. *Conclusions.*

This brings us to one of the objects of the whole discussion. Let me put it in the form of a simple question. Do positive and negative numbers constitute a complex number system of two units $e_0=+1$, $e_1=-1$, with the domain of ordinary arithmetical rational and irrational numbers for the coordinate range, or do they not constitute a complex system, and, if not, why not? Surely $+1$ and -1 are units and are qualitatively distinct. According to the second definition, then, we may define $a_1(+1) + a_2(-1)$ as a complex number of two units. But in this algebra we know that $a_1(+1) + a_2(-1) = (a_1 - a_2)(+1)$ when $a_1 > a_2$; $= 0$, when $a_1 = a_2$; and $= (a_2 - a_1)(-1)$ when $a_2 > a_1$. Hence these qualitative units are such that though themselves distinct, when coordinates are attached, we arrive at expressions which are not unique. Thus if $(a_1, a_2) = (b_1, b_2)$, we may infer $a_1 = b_1, a_2 = b_2$; and if $(a_1e_1 + a_2e_2) = (b_1e_1 + b_2e_2)$, then $a_1 = b_1$, $a_2 = b_2$. But if $a_1(+1) + a_2(-1) = b_1(+1) + b_2(-1)$, the most we can infer is that $a_1 = b_1 \pm b$, $a_2 = b_2 \pm b$. We meet the same problem in operators and simply evade it by defining $(+1)\alpha + (-1)\alpha = 0\alpha$. The question still remains.

If an n-tuple may reduce to an $(n-m)$-tuple it ceases to be properly an n-tuple. Hence the n-tuple algebra must exclude the positive-and-negative-number calculus as not an algebra. So also if the n-manifold may have its number of dimensions reduced, it ceases to be an n-manifold. So this calculus must be excluded here. A vector algebra of positive and negative quantities is one-dimensional and not properly a vector algebra at all. In fact, the n-tuple and the manifold exclude any possibility of ever reducing the number of positions or of units by additions of their coordinates which belong to different positions or different units. Technically, the units are linearly independent so far as entities of the range are concerned. Now $+1$, -1, are not linearly independent so far as the domain of the integer or the real number are concerned. Thus, for example $2(+1) + 1(-1) = 1(+1)$.

The third definition places the difficulty farther back. Thus $(+1)$ and (-1) are independent or not according as $(+1)\,a$ and $(-1)\,a$, where a is the operand, are independent or not.

The fourth definition admits such lack of independence, for we define -1 by this very lack of independence. Indeed, we define -1 to be the entity x such that

$$x + 1 = 0 \quad \text{or} \quad x + a = a - 1.$$

That is, we work modulo $x + 1$. We extend our old arithmetical addition to a new ideal range. On the basis of the fourth definition, positive and negative numbers do constitute an algebra of two distinct qualitative units, but not a two-dimensional algebra. In fact, we may assert at once that the fourth definition defines ideal numbers which are qualitatively distinct, but which may not be able to be ordered as many-dimensional. We do not at first know whether the different ideals are many-dimensional or not. Thus $q^2 - 2wq + t^2 = 0$ is satisfied by any number of sets of quaternions, and the dimensions may thus become infinite.

The distinction may be illustrated by the difference between a line in a two-dimensional bounded non-euclidean space and such space itself. The portion of the line external to the space may be called ideal, but the real and the ideal (qualitatively different) portions of the line are yet one-dimensional.

Modifications of the usual theory of algebras are necessary in these algebras with, we may say, deficiency of dimension. Thus for positive and negative numbers we do not have a characteristic equation of order two, which we should expect, but simply the equation

$$(+1) + (-1) = 0.$$

Another example occurs in ordinary complex numbers, which constitute an algebra of four qualitative units, e_0, e_1, e_1^2, e_1^3, with the equations $e_0 + e_1^2 = 0 = e_1 + e_1^3$. The characteristic equation is not of order four, but for $\rho = xe_0 + ye_1 + ze_1^2 + we_1^3$ is $(\rho - x + z)^2 + (y - w)^2 = 0$.

As another example we might define an algebra by working modulo $e_1^2 - 1$, $e_2^3 - 1$, $e_1e_2 - e_2^2e_1$, $e_2 + e_2^2 + 1$, $e_1e_2 + e_1e_2^2 + e_1$.

These examples show the great power and generality of the fourth definition. It is evident that we define an algebra thus by what may be called its invariant equations.

The criticism made by Russell, that algebraic definitions do not guarantee the existence of the thing defined, does not apply to this method of putting the definition. If algebra exists, then we may work modulo $f(q)$ if we choose, and $f(q)$ is simply an expression in the algebra. We thus arrive at expressions containing q. If we choose to call these hypercomplex numbers, we do not thereby destroy them. If under given conditions it turns out that we may work modulo a whole infinity of expressions $f(q_1)$, $f(q_2)$, ..., then we have simply found a kingdom where we searched for much less.

April, 1906.

HOW SHOULD THE COLLEGE TEACH ANALYTIC GEOMETRY?*

BY PROFESSOR HENRY S. WHITE.

In most American colleges, analytic geometry is an elective study. This fact, and the underlying causes of this fact, explain the lack of uniformity in content or method of teaching this subject in different institutions. Of many widely divergent types of courses offered, a few are likely to survive, each because it is best fitted for some one purpose. It is here my purpose to advocate for the college of liberal arts a course that shall draw more largely from projective geometry than do most of our recent college text-books. This involves a restriction of the tendency, now quite prevalent, to devote much attention at the outset to miscellaneous graphs of purely statistical nature or of physical significance. As to the subject matter used in a first semester, one may formulate the question: Shall we teach in one semester a few facts about a wide variety of curves, or a wide variety of propositions about conics?

Of these alternatives the latter is to be preferred, for the educational value of a subject is found less in its extension than in its intension; less in the multiplicity of its parts than in their unification through a few fundamental or climactic principles. Some teachers advocate the study of a wider variety of curves, either for the sake of correlation with physical sciences, or in order to emphasize what they term the analytic method. As to the first, there is a very evident danger of dissipating the student's energy; while to the second it may be replied that there is no one general method which can be taught, but many particular and ingenious methods for special problems, whose resemblances may be understood only when the problems have been mastered. No one can lay down rules *a priori* which would enable the student to rediscover the theorems now known concerning conics — if these were once lost. Were there such a single well-defined method or art of creating geometry, should we still be waiting for the completion of the theory of plane quartics, or for even an outline of the theory of quintics? It is rather the art of asking questions that is to be taught. How can one learn what kind of questions will prove fruitful, save by examining those which have been asked and answered by geometricians in the past?

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* Read before the Association of Teachers of Mathematics of the Middle States and Maryland, at Annapolis, Md., December 1, 1905.

If the hundreds of important theorems in conics were really unrelated, the argument for their study simply as pertaining to conics would be weak. But just there lies the point; modern geometry has found general principles which unify the whole subject. Every part is connected with every other part. The student learns to employ many methods of inquiry or of proof, but does not lose himself in endless diversity. And while no one can affirm that a branch of science is complete, yet all historians point to the theory of conics as a model both in the large number of its particular propositions and in the possession of general concepts and theorems. It is the most optimistic dream of workers in other branches of mathematics that their favorite special subject may some day reach a stage of development equally advanced with that to which two thousand years have brought this ancient favorite of geometricians.

Conics is an ample range for one semester, and needs but little enlargement for a second. Of the methods to be employed not much need be said to experienced teachers. For the first two weeks, constant drill work on points, distances, areas, circles, and parabolas. By that time the notion of a language for definite use will impress itself. The desire to express a distance in rectangular coördinates calls up a radical covering the sum of two squares, and conversely the sum of two squares of differences of like coördinates recalls involuntarily the motion of a distance; the equation of a circle comes to look to the eye like the definition of a circle in words. A parabola is thought of when an equation separates into one linear part and one square of a different linear expression. Proportional division becomes as familiar as the inflection of a Latin noun, and the area of a triangle in terms of vertices, a string of six terms, comes to look like a moderately long German word. Analysis and synthesis have begun to be instinctive, as they are when one has taken the first steps in a foreign language.

Next comes the patient extension of the vocabulary simultaneously with the study of new geometrical notions. Slope, intercepts, systems of lines, pencils of conics prove easy one after another, and the first milestone is reached when after twenty hours of class-room work the student perceives that every linear expression denotes a point-function with niveau lines straight and parallel, and that all such expressions may serve as coördinates equally well with his favorite x and y.

This is the point from which geometry should be the main object, and the properties of conics should be developed. Of course metrical properties must come first. The parabola with its axis, focus, vertex, and directrix must be fixed by drawing practice; the bisection of the sub-tangent by the vertex will naturally be the first theorem, and the constancy of the subnormal the next. Mutually perpendicular tangents meeting on the directrix, and circles circumscribed to circumscribed triangles are important enough to demand attention from even the most hurried student. The diameters of parabolas and of all other conics should be studied together, and conjugate pairs of diameters with their use as axes of reference.

So far, probably every teacher will agree, the student's attention should be confined to metric properties; and indeed in view of the number, the beauty, the practical value of metric properties of conics, it is not surprising that most American text-books include nothing more general. But it is to be remembered that the aspect of this large body of truths to the beginner is not the same as to the more advanced student or teacher. The former is first pleased with the novelty, then wearied with the multiplicity of disconnected facts, and will soon become bewildered and discouraged. A passage from Taylor's Ancient and modern geometry of conics puts the case so well that I will quote it here. "The mind of the tyro is commonly overwhelmed with a multitude of details not reduced to any system, demonstrations are put before him in an unsuggestive form which gives no play to his inventive faculty; and thus it comes to pass that of the many students so few turn out genuine geometers. Let the learner be furnished with principles, and not alone with fully explained facts, and be continually stimulated to exertion by the intense pleasure of finding something left to discover for himself." (P. LXXVII, § 4.) In contrast to this attitude of the neophyte confronted with the wealth of existing theorems of metric character, let any teacher, familiar with projective geometry, review Pascal's chapter (Repertorium der höheren Mathematik, part II) or any other brief synopsis of this subject. He will note the rising tide of pleasure and admiration as well-known and less known discoveries succeed one another, in all some sixty or seventy facts of major importance; and will probably observe that half of his satisfaction is referable to his serene conviction that all these things flow as easy deductions from a very small number of projective theorems.

Projective geometry is acknowledged to furnish a fascinating continuation or climax to metric geometry; but if it can also serve as a clue, a map amid the intricacies of the elementary theories, why should it not be employed for that purpose? After working two-thirds of a semester to acquire an understanding of the objects and a preliminary knowledge of the theorems of metric conics, let the student learn the few fundamental and comprehensive projective theorems. First of all comes, of course, the relation of conjugate points, commonly stated in the form: If a point P lies on the polar of a point Q, then conversely Q will lie on the polar of P. Without much assistance the student can see that this theorem contains no reference to axes of coordinates, and will suspect that it points to a more general geometry than he has yet studied. Next he can be made acquainted with the harmonic position of four points, or with the anharmonic ratio of any four collinear points or copunctual lines, and then return for practice to the theorems on conjugate diameters. Then he should learn that the anharmonic aspect of four points on a conic is the same from all points of that conic, and the dual theorem on tangents. He has now in hand the projective (organic) generation of the general conic, the unifier of many facts previously independent.

Returning to the metric basis, I would then establish the use of trilinear coordinates by the use of the formula for distance of point from line. Thereupon one may present Salmon's proofs for Carnot's theorem concerning the segments cut by a conic on the sides of a triangle; and if the theorem of Menelaus on the straight transversal of a triangle has been introduced earlier, Pascal's hexagon may be made the terminus of the first semester's study. Less than this ought not to satisfy us. As no college student of chemistry ought to be a whole semester in ignorance of Mendeléeff's table; as no college student of physics should pass an examination before learning, however crudely, some reasons for believing in the conservation of energy; so in analytic geometry let us give even the beginner in his first course some knowledge of Pascal's immortal theorem.

In the college course, the average student of sciences cannot afford time for more than one semester of geometry, even though the analytic auxiliaries may seem to open a whole new world to his vision. He must hasten to master the elements of the differential and integral calculus and attack applied mathematics. Hence the necessity of making the first semester of analytic geometry less slow and richer in material than is at present customary. But provision ought to be made for another year's work for those who wish to become mathematicians. Their second semester should make them acquainted first with quadric surfaces, both as individual surfaces and as associated in a pencil and in a confocal system, but next and by no means less in importance, with the remaining chief propositions concerning plane conics. These are perhaps those of Desargues and Steiner on two perspective triangles, that of Brianchon on the sextette of tangents, with modifications for quintettes, quartettes, and triads; Hesse's famous theorem on the six vertices of a complete quadrilateral, four of which form two conjugate pairs in a polarity, and the theorems of Poncelet concerning polygonal lines inscribed to one conic and circumscribed to another. And lack of famous names should not cause us to omit those relating to pairs of triangles, the first where the triangles are polar reciprocals with respect to a conic, and the second where each is self-polar.

These latter theorems, with due attention to the powerful methods of projection and reciprocation, may well enough fill up the second semester; of course each one should be the occasion of two or three careful exercises in drawing and in schematic constructions. The third semester will find its most appropriate material in mutually apolar systems of conics, and in a rapid discussion of plane cubic curves. At present this is, so to speak, the end of improved roads in one direction, and students who come so far may easily obtain such outlooks as the ambitious are sure to desire; and they will have sufficient experience of travel to warrant them in undertaking explorations upon their own account. Apropos of this we may notice Loria's summary of the present situation, in his valuable compend of Special plane curves (pages 219–20).

"The theory of curves of the third order has clearly become, so to speak, a province of the realm of mathematics; the theory of curves of the fourth order is a territory only partly subjected to this kingdom; but the theory of curves of any particular order higher than the fourth still asserts its independence and resists stubbornly every attempt to introduce law and order. If we except [certain special researches upon quintic curves] we may say that up to the present nothing has occurred to illuminate the thick darkness which conceals from us the properties of those curves which should come next after conics, cubics and quartics. Not only so, but there is not even any special curve of fifth order which has received a particular name."

The course here advocated is somewhat modernized from the traditional type of college courses in geometry, but it is not, therefore, chimerical. A survey of the field should convince us that in three semesters a college student can become posted on the essentials of conics, quadric surfaces, and cubic curves. Individual teachers will prefer a different order from that here advocated, either for theoretical reasons or on account of the special qualifications of the students under their charge; but it is believed that the main thesis will command general approval — the most modern methods and the most general propositions of present day geometry must be made available for the college student.

VASSAR COLLEGE,
November 29, 1905.

FOUR BOOKS ON THE CALCULUS.

Die Anfangsgründe der Differentialrechnung und Integralrechnung. Für Schüler von höheren Lehranstalten und Fachschulen sowie zum Selbstunterricht. Dargestellt von DR. RICHARD SCHRÖDER. With numerous exercises and 27 figures in the text. Leipzig, B. G. Teubner, 1905. vii + 131 pp.

Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen. Zusammengestellt von DR. ROBERT FRICKE. Fourth edition, with 74 figures in the text. Braunschweig, Friedrich Vieweg und Sohn. xv + 217 pp.

Repetitorium und Aufgabensammlung. Von DR. FRIEDRICH JUNKER. Second, improved edition, with 46 figures in the text. Leipzig, G. J. Göschen, 1905. 16mo. 127 pp.

Sammlung von Formeln und Sätzen aus dem Gebiete der elliptischen Functionen, nebst Anwendungen. Von J. THOMAE. Leipzig, B. G. Teubner. 43 pp.

THESE four titles are here grouped for the purpose of making certain comparisons both among themselves and with current American texts in the corresponding fields.

The first volume, as its title indicates, is an introduction to the calculus and is intended for use in the upper classes of gymnasia, in accordance with the permission afforded in the Prussian curriculum of 1901. While the text is strictly elementary, it is clear and logical in its development, beginning with limits notions and geometric interpretation of the derivative. The derivatives of all ordinary forms, including inverse and implicit functions, are deduced before any exercises are introduced; then over one hundred exercises, very many of which are complicated combinations of algebraic and transcendental functions, are given in one list, with the results in each case printed in juxtaposition. Eleven pages, or one-twelfth of the book, are devoted to indeterminate forms, with illustrations but no exercises. Thirty-three pages are given to tangents, normals, subtangents, subnormals, etc., the forms of which are all worked out for the circle, parabola, ellipse, hyperbola, semicubical parabola, cissoid, cycloid, lemniscate and cardioid, with no exercises left for the student. Likewise rectifications and quadratures for these curves are worked out in full in the text.

The second volume is the fourth edition of Fricke's well-known text, improved and enlarged. When we consider the clearness and rigor of presentation and contemplate the number of subjects here treated within a little more than two hundred pages, including all the ordinary topics of elementary calculus, together with chapters on the theory of definite integrals, ordinary differential equations, and functions of a complex variable, we wonder how it can be done in so small a space. The explanation is that the book contains very few exercises for the student and the development of principles goes on uninterruptedly from beginning to end. From the standpoint of the purely theoretical science, this presentation of the subject has its advantage in that the attention is not diverted from the steady swing of the doctrine from topic to topic; and the student whose mind quickly and readily grasps the principles of the calculus in the abstract must surely find great delight in the clear and vigorous presentation, the pleasing arrangement of the page, and the skillful development step by step, from the elementary to the more involved theorems. But the student who must handle the concrete before he grasps the abstract, who must work through many exercises up to the comprehension of the principles, will need to combine with such a text as Fricke's a good collection of exercises and applications in the differential and integral calculus.

Fortunately such exercise books are not wanting in Germany, and the third title in the list above is a book of this kind for the differential calculus. It is a small volume containing nearly five hundred exercises, more than half of which are set forms for differentiation or for the formal application of the rules under the various topics, and in all cases the results are given with the exercises, leaving no chance for independent discovery. Of the remaining exercises which are more in the nature of problems, all are geometric, relating to maxima and minima, asymptotes, tangents, normals and numerous properties of curves and surfaces, and under all the topics large numbers of illustrations are worked out in full. There are no application to mechanics or other subjects.

These two volumes of Fricke and Junker (with a corresponding exercise book on the integral calculus) would seem to constitute a somewhat typical provision for the needs of the German student of the calculus, quite in contrast with the plan of our own text-books — on this and other subjects. In favor of the former plan is the greater compactness of the text and the more closely knit continuity in the development of the principles; while the latter plan secures the juxtaposition of principles and applications in a single volume and seems more certain to secure careful attention to the concrete side of the work on the part of the student.

The last title here under consideration represents still another scheme of arrangement and presentation of formulas and principles of a subject, quite in contrast with the other three volumes and also with publications in this country. Thomae has collected the chief formulas and theorems in the realm of elliptic functions, without proofs or explanations beyond what is necessary for a clear understanding of the notation. These are included compactly but clearly under forty-two divisions covering only twenty-six pages and the remaining eighteen pages are devoted to brief expositions of twenty typical applications of the elliptic functions. The collection of formulas is based on the Jacobi-Legendre fundamental functions and notation, as being, in the author's judgment, better adapted to applications involving numerical computation than the Weierstrassian forms used by Schwarz in his collection, though these latter are of the highest theoretical importance.

Thomae's collection puts in concise and accessible form the whole range of the elliptic function doctrine, as based on the theta function, the zeta function, the omega function, and the Legendre normal forms, and shows fully how useful and practical this development becomes in application to a wide class of problems.

In conclusion it seems quite clear that collections of principles and formulas are highly appropriate and useful for students in advanced stages of progress, but that for elementary students the form of presentation to be commended is that in which problems and exercises are skilfully used in the text to lead up to the statement and proof of principles, as well as to illustrate and clarify the theory in immediate connection with its formal development.

H. E. SLAUGHT.

THE UNIVERSITY OF CHICAGO,
May 28, 1906.

SHORTER NOTICES.

Einleitung in die Funktionentheorie. Abteilung 2. By OTTO STOLZ and J. ANTON GMEINER. Leipzig, B. G. Teubner, 1905. viii + 355 pp.

WITH the publication of the second part of the Stolz and Gmeiner Funktionentheorie, the revision of Stolz's Allgemeine Arithmetik is complete. The Theoretische Arithmetik and the Funktionentheorie, which must still be regarded as parts of the same whole, together present a course in analysis which begins with the integers and includes all the usual operations except differentiation and integration. Taken in connection with Stolz's Calculus, they form a kind of German Cours d'Analyse.

The volume before us begins with Chapter VI, containing a thorough discussion of convergence and divergence criteria for infinite series. The chapter is new and, according to the preface, due to Gmeiner, who has based it principally on the article of Pringsheim in the thirty-fifth volume of the *Mathematische Annalen*. The subject is an extremely interesting one, connecting with several of the most difficult problems in analysis. Its connection with the related problem of determining scales to represent the orders of infinity of functions is fairly evident though not emphasized; but its equivalence with the problem of the convergence of

$$\int_a^\infty f(x)dx = \mathrm{L}_{b=\infty} \int_a^b f(x)dx$$

we did not find referred to, doubtless because the book does not take up the theory of definite integrals.

Chapter VI is somewhat in the nature of a digression since it comes between a chapter on power series and Chapter VII, the theory of monogenic analytic functions. The last mentioned chapter is based chiefly on lectures of Weierstrass and was not contained in the Allgemeine Arithmetik. It is followed by a chapter on the circular functions which contains in more extended and detailed form the contents of chapter 6, volume II of the older work, *i. e.*, the theory of the functions a^x, $\log x$, $\sin x$, $\sin^{-1}x$, etc., for complex values of x.

Chapters IX, X, and XI have all been revised and enlarged by Gmeiner. The chapter on infinite products (IX) differs from the corresponding chapter of the Allgemeine Arithmetik in that the series formed by taking the logarithms of the factors is used as little as possible. There have been added a number of theorems due to Arzelà about products whose factors are functions of an independent variable, thus involving the notion of uniform convergence. The treatment of finite and infinite continued fractions (X, XI) has also been much extended by Gmeiner. Among other changes, such expressions as

$$3 + \frac{2}{7} + \frac{5}{2} - \frac{1}{1} - \frac{1}{2},$$

which in the Allgemeine Arithmetik were regarded as "sinnlos "* are here designated as "improper" continued fractions and their value determined by the following definition: The expression

$$(1) \qquad b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}$$

determines the quantities Z_k by the equations

$$(2) \qquad \begin{gathered} Z_n = b_n, \quad Z_{n+1} = 1, \\ Z_m = b_m Z_{m+1} + a_{m+1} Z_{m+2} \\ (m = n-1, n-2, \cdots, 1, 0). \end{gathered}$$

The value of (1) is Z_0/Z_1. The only fractions which now fail to have values are those for which Z_1 is zero. If $Z_\kappa = 0$ for any value of κ other than 1, the fraction is improper. This definition leads to no new extension of the notion of an infinite continued fraction because improper convergents had been used previously even where the corresponding continued fractions were regarded as meaningless. Among the modern developments added to the chapter on infinite continued fractions, it is interesting to note the theorems of Van Vleck published in the second volume of the *Transactions*.

To sum up this notice and also the review of the first volume in last December's BULLETIN — the work as a whole gives an effect of conservatism, maturity and poise. It is not as likely as the modern French books to stimulate research, but it has a permanent value as a repository of accurate information about the conventional functions and processes of analysis.

OSWALD VEBLEN.

Lezioni sul Calcolo degli Infinitesimi, date nella R. Università di Modena da ETTORE BORTOLOTTI. Raccolte dal Dr. ARMANDO BARBIERI. Modena, 1905. vii + 61 pp.

BORTOLOTTI's Calcolo degli infinitesimi is a short course of lectures on various questions that arise in connection with the determination of the relative orders of two infinitesimals (infinites)

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*Because $2 - \frac{1}{1} - \frac{1}{2} = 0$. The significance of this definition of "improper" continued fractions may perhaps be suggested by the equation

$$3 + \frac{2}{7} + \frac{5}{0} = 3 + \frac{2 \cdot 0}{0 \cdot 7 + 5} = 3.$$

$f(x)$ and $\phi(x)$, and consequently includes a discussion of the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, etc. The book is intended for comparatively young students. It begins by defining the upper and lower bounds of indetermination of $\psi(x)$ as x approaches a limiting value and recapitulating some general theorems on limits. It then defines the equality of orders of $f(x)$ and $\phi(x)$ as meaning that $|f(x)/\phi(x)|$ has finite and positive bounds of indetermination. If the limit of $f(x)/\phi(x)$ exists and is infinite as x approaches a limiting value, $f(x)$ is said to be an infinitesimal of lower order than $\phi(x)$; $f(x)$ is of order m if of the same order as $(x-a)^m$.

The difficulties of this definition, for example that functions exist whose order is less than $m+\alpha$ and greater than $m-\alpha$ for every $\alpha > 0$ without being equal to m, are exhibited and used to lead up to Cauchy's definition: $f(x)$ is infinitesimal of the same order as $\phi(x)$ if for every constant $\alpha > 0$,

$$\lim_{x=a} \frac{f(x)}{[\phi(x)]^{1+\alpha}} = \infty, \lim_{x=a} \frac{f(x)}{[\phi(x)]^{1-\alpha}} = 0.$$

This, as Bortolotti points out, is equivalent to saying that when $\epsilon(x)$ is defined by the equation $f(x)/\phi(x) = [\phi(x)]^{\epsilon(x)}$,

$$\lim_{x=a} \epsilon(x) = 0.$$

This definition is subject to the inconvenience of requiring two functions to be of the same order in some cases where their ratio approaches the limit ∞, but it has the advantage of extending the notion of order to a wider class of functions than is reached by the first definition.

After developing the properties of Cauchy's definition of order, Bortolotti goes on to expound in a clear and rigorous way the rules derived by l'Hôpital, Cauchy and Stolz for the evaluation of the indeterminate forms $0/0$ and ∞/∞. Here, it is to be remarked that to the hypothesis of the theorem on page 25 should be added the sentence, "If $\phi(x)$ is infinitesimal, so is also $f(x)$."

These more or less familiar theorems are followed by methods of determining the relative orders of functions by examining the double ratios

$$\frac{f'(x)}{\phi'(x)} : \frac{f(x)}{\phi(x)}, \quad \frac{\Delta f}{\Delta \phi} : \frac{f}{\phi}.$$

Some of these criteria might perhaps have been stated to better advantage in terms of the derivatives of the logarithms of $f(x)$ and $\phi(x)$. The book ends with a consideration of the indeterminate forms $\infty - \infty$, 1^∞, etc.

Taken as a whole, the book is not only clear, readable, and fairly thorough, but has a good set of references to other works and a rich list of problems which may be useful in elementary calculus courses. It therefore seems to deserve a place in any fairly complete mathematical library.

OSWALD VEBLEN.

Abstrakte Geometrie. By KARL THEODOR VAHLEN. Leipzig, B. G. Teubner, 1905. xi + 302 pp.

A book following the general outline of Vahlen's Abstrakte Geometrie would be very useful for giving a general view of the recent studies on foundations of mathematics. This is sufficiently indicated by the titles of the chapters: I, Foundations of arithmetic; II, Projective geometry (theorems of connection); III, Projective geometry (theorems of order); IV, Affine geometry (euclidean and non-euclidean); V, Metric geometry.

Unfortunately, however, the book is not characterized by that precision of language which is indispensable in any discussion of such a subject. The reader is constantly confronted with statements which are incorrect if taken literally and which, if not taken literally, are open to more than one interpretation. Many of the author's postulates are labeled by him as definitions. Moreover, there are places where it is very difficult to determine which of the previously stated hypotheses are being used and which are not. As a consequence, the reviewer is able to state hardly a single new *result* which is surely established by this book.

On the other hand, there are many *suggestions* of methods which if rigorously carried out would probably lead to interesting and elegant results. For example, the notion of planar order is defined not by means of coördinates as in the usual analysis, nor by the way in which a straight line intersects a triangle (according to Pasch) but by means of postulates in terms of the right and left sides of a line with respect to a given sense on the line. One is thus enabled to deal at once with the most general type of planar-ordered set without presupposing anything about a plane in which it lies.

For suggestions such as the one mentioned, the book may be recommended to anyone who is already familiar with other works on the foundations of geometry, but for a beginner it would be thoroughly misleading, and to a philosophical outsider who wished to learn the methods and ideas of the logic of mathematics it would give some very queer notions indeed.

It seems to the reviewer not to be worth while to lengthen this notice with criticisms of details, especially as many of the points that would be mentioned have already been adverted to by Dehn in a review published in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 14, page 535. The reader who is interested in such things will find a rejoinder to Dehn by Vahlen on page 591 of the same volume, a retort by Dehn on page 595, and a second "Erwiderung" by Vahlen in volume 15, page 73. With these he may compare a footnote by Schoenflies on page 31, volume 15.

OSWALD VEBLEN.

Quadratic Partitions. By Lt. Col. ALLAN CUNNINGHAM, R.E. London, Francis Hodgson, 1904. xxiii + 266 pp.

THE main tables in the book under review contain the quadratic parts (t, u) of the partitions

$$p = t^2 \pm Du^2,$$

wherever possible, for all values of $D < 20$ $(D \neq k^2 \cdot \delta)$, and for all primes p to various limits not above 100,000. Shorter tables at the end of the book contain solutions of the Pellian equations

$$\tau^2 \pm Dv^2 = \pm 1, \quad \pm 2, \quad \pm 4, \quad \pm 8, \quad \pm 16$$

for various values of D not over 1,000.

In the Introduction the author gives a brief, but excellent sketch of the properties of quadratic forms, and describes methods of applying the present tables to factorizations, the calculation of Hauptexponenten and kindred problems.

Comparing it with the earlier tables of Jacobi and Reuschle, the student of number theory cannot fail to be impressed with the excellence of Col. Cunningham's book. Neatness, freedom from errors, and admirably compact arrangement of tables of such extent add to their appearance as well as their usefulness. The odd primes are printed *forty* on each page, an arrangement which greatly facilitates counting the primes that fall in a given interval, and reduces the liability to error in computations where a number of successive primes are used.

In addition to errata noted by the author, an obvious misprint occurs in the Introduction: page ix, line 4 from bottom, for $(3\tau^2 v + qv^3)$ read $(3\tau^2 v + Dv^3)^2$.

J. C. MOREHEAD.

ERRATA.

The following errata in the present volume of the BULLETIN have come to the attention of the editors :

Page 74, line 28, *for* k *read* $k > 1$.
Page 75, line 38, *for* $3^{k+\beta}$ *read* $2^{k+\beta}$.
line 42, *for* 0, 1 *read* $0, n$; $n > 0$.

NOTES.

AT the meeting of the London mathematical society, held on May 10, the following papers were read: By B. RUSSELL, "On the substitution theory of classes and relations"; by E. CUNNINGHAM, "On linear differential equations of rank unity"; by E. J. ROUTH, "On the motion of a swarm of particles whose center of gravity describes an elliptic orbit of small eccentricity about the sun"; by H. BATEMAN, "The theory of integral equations"; by G. H. HARDY, "Singularities of power series in two variables."

THE appearance of the April number (volume 28, number 2) of the *American Journal of Mathematics* has been delayed by an extensive printers' strike in Baltimore.

THE University of Kiel announces the following prize problem for the year 1906–1907 :

It is required to determine the relations which exist between the principal integrals of the various regions, for the hypergeometric differential equation of the third order with two finite singular points.

THE following advanced courses in mathematics are offered during the year 1906–1907 by the universities named below :

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Projective geometry, two hours; Dynamics, two hours (first half year); Theory of functions, two hours (second half year); Classic authors, one hour. — By Dr. A. COHEN: Elementary theory of functions, two hours; Differential equations, three hours (first half year); Theory of numbers, three hours (second half year). — By Dr. A. B. COBLE: Theory of correspondence, two hours.

INDIANA UNIVERSITY. — By Professor R. J. ALEY: Differential equations, five hours; Theory of numbers, six hours. — By Professor S. C. DAVISSON: Modern analytic geometry, four hours; Theory of surfaces, four hours; Fourier's series, three hours. — By Professor D. A. ROTHROCK: Calculus, II, six hours; Calculus of variations, six hours; Functions defined by differential equations, four hours. — By Professor U. S. HANNA: Groups of substitutions, three hours; Galois's theory of equations, three hours.

(Summer term, June 21–September 7, 1906). — By Professor S. C. DAVISSON: Higher plane curves, five hours. — By Professor D. A. ROTHROCK: Calculus of variations, six hours. — By Professor U. S. HANNA: Theory of numbers, three hours.

UNIVERSITY OF WISCONSIN (Summer session). — By Professor C. S. SLICHTER: History of mathematics, two hours; Differential equations, five hours. — By Professor G. A. BLISS: Elliptic functions in the Jacobi form, five hours; Calculus of variations, three hours.

PROFESSORS G. CASTELNUOVO and V. CERRUTI, of the University of Rome, and Professor A. CAPELLI, of the University of Naples, have been elected to membership in the royal institute of Lombardy.

DR. P. CALAPSO has been appointed docent of analytic and projective geometry at the University of Palermo. Dr. G. MARLETTA was appointed to a similar position at the University of Catania.

PROFESSOR D. v. STERNECK has been promoted to a full professorship of mathematics at the University of Czernowitz.

Dr. A. Hasenöhrl, of the University of Vienna, has been appointed associate professor of physics at the technical school of Vienna.

Professor N. N. Saltykow, of the technical school at Kiew, has been appointed associate professor of pure mathematics at the University of Charkow.

Professor W. A. Stekloff, of the University of Charkow, has been appointed professor of applied mathematics at the University of St. Petersburg.

Dr. K. Wieghardt, of the technical school at Aachen, has been appointed associate professor of mechanics at the technical school at Brunswick.

Dr. E. Schmid has been appointed docent in mathematics at the University of Bonn, and Dr. Schellfisch to a similar position at the University of Münster.

Professor E. W. Brown, of Haverford College, has been appointed to a professorship of mathematics at Yale University. Professor Brown will remain at Haverford until the autumn of 1907.

Dr. R. B. Allen, of Clark University, has been appointed professor of mathematics at Kenyon College, Gambier, Ohio.

At Columbia University, Dr. Edward Kasner has been promoted to an associate professorship of mathematics.

Dr. H. F. Stecker has been promoted to an assistant professorship of mathematics at the Pennsylvania State College.

At Harvard University, Mr. J. K. Whittemore has been promoted to an assistant professorship of mathematics.

Mr. A. S. Eve has been appointed assistant professor of mathematics in McGill University.

Mr. F. R. Sharpe, Dr. Arthur Ranum, and Dr. W. B. Carver have been appointed instructors in mathematics at Cornell University.

Dr. R. L. Moore has been appointed instructor in mathematics at Princeton University.

Mr. W. M. Thomas has been appointed tutor in mathematics in the College of the City of New York.

MISS M. M. YOUNG has been appointed instructor in mathematics in Wellesley College.

DR. ROXANA H. VIVIAN, instructor in mathematics at Wellesley College, will be a member of the faculty of the American college for girls in Constantinople the coming year, on leave of absence from Wellesley.

PROFESSOR GUSTAV BAUER, of the University of Munich, died April 3, at the age of 86 years.

PROFESSOR G. OLTRAMARE, emeritus professor of the University of Geneva, died April 10 at the age of 90 years.

PROFESSOR G. A. WENTWORTH, formerly of Phillips Exeter Academy, died May 24, at the age of 71 years. Professor Wentworth had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1891.

Catalogues of second hand mathematical works: Gustav Fock, 7 Schlossgasse, Leipzig, catalogue no. 280, 2802 titles in mathematics, astronomy, geology; W. Junk, 201 Kurfürstendamm, Berlin, catalogue no. 28, about 2400 titles in astronomy, physics, and mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CLEBSCH (A.). Vorlesungen über Geometrie, bearbeitet und herausgegeben von F. Lindemann. (In 2 Bänden.) 1ter Band: Geometrie der Ebene. 1ter Teil: Kegelschnitte und algebraische Formen. 1te Lieferung. 2te, vermehrte Auflage. Leipzig, Teubner, 1906. 8vo. 480 pp. M. 12.00

FABER (G.). Ueber die zusammengehörigen Konvergenzradien von Potenzreihen mehrerer Veränderlicher. Karlsruhe, 1905. 8vo. 36 pp.

FISHER (I.). A brief introduction to the infinitesimal calculus; designed especially to aid in reading mathematical economics and statistics. New edition. New York, Macmillan, 1906. 12mo. 13 + 84 pp. Cloth. $0.75

GARBIERI (G.). Complementi di algebra: riassunto di lezioni date nelle università di Padova e di Genova. Torino, Paravia, 1906. 8vo. 32 pp. L. 2.00

HAGEN (J. G.). Synopsis der höheren Mathematik. Band III: Differential- und Integralrechnung. 7te Lieferung. Berlin, Dames, 1905. 4to. 6 + pp. 385–471. M. 5.00

Hess (A.). Stetige Abbildung einer Linie auf ein Quadrat. Zürich, 1905. 8vo. 45 pp. M. 2.00

Klein (F.). Autographierte Vorlesungshefte. Heft III: Ueber die hypergeometrische Funktion. Vorlesung, gehalten im Wintersemester 1893–94. Ausgearbeitet von E. Ritter. Neuer unveränderter Abdruck. 4 + 568 pp. M. 9.00

——. Heft V: Riemannsche Flächen. Ite Vorlesung, gehalten während des Wintersemesters 1891–92. Neuer unveränderter Abdruck. 4 + 301 pp. IIte Vorlesung, gehalten während des Sommersemesters 1892. Neuer unveränderter Abdruck. 4 + 288 pp. Leipzig, Teubner, 1906. M. 1 .00

Lebesgue (H.). Leçons sur les séries trigonométriques, professées au Collège de France. Paris, Gauthier-Villars, 1906. 8vo. 136 pp. Fr. 3.50

Lebon (E.). Tables de caractéristiques relatives à la base 2,310 des facteurs premiers d'un nombre inférieur à 30,030. Paris, Delalain, 1906. 8vo. 32 pp. Fr. 1.50

Lévy. See Rouché.

Lindemann (F.). See Clebsch (A.).

Lindow (M.). See Sohncke (L. A.).

Manning (H. P.). Irrational numbers and their representation by sequences and series. New York, Wiley, 1906. 12mo. 6 + 123 pp. Cloth. $1.25

Mattson (R.). Contributions à la théorie des fonctions entières. Upsala, 1905. 8vo. 97 pp. M. 2.50

Ritter (E.). See Klein (F.).

Rouché et Lévy. Analyse infinitésimale, à l'usage des ingénieurs. Tome 2. Paris, Gauthier-Villars, 1905. 8vo. 7 + 648 pp. Fr. 15.00

Sohncke (L. A.). Sammlung von Aufgaben aus der Differential- und Integralrechnung. Teil II. 2te Abteilung: Integralrechnung. II. 6te verbesserte Auflage. Bearbeitet und herausgegeben von M. Lindow. Jena, Schmidt, 1906. 8vo. 6 + 224 pp. M. 5.00

Vallée Poussin (C. J. de la). Cours d'analyse infinitésimale. Tome II. Louvain, 1906. 8vo. 16 + 440 pp. Fr. 12.00

Veronese (G.). Il vero nella matematica. Roma, Forzani, 1906. 8vo. 38 pp.

II. ELEMENTARY MATHEMATICS.

Blomfield (C. H.). See Jones (A. C.).

Cracknell (A. G.). See Workman (W. P.).

Deakin (R.). Euclid, Books I, III. Preliminary certificate edition (for course B). London, Clive, 1906. 8vo. 296 pp. Cloth. 2s. 6d.

D'Ovidio (E.). See Sannia (A.).

Eggar (W. D.). Manual of geometry. London, Macmillan, 1906. 8vo. 350 pp. Cloth. 3s. 6d.

Fernández de Prado (G.). Manual de la regla de cálculo. Exposición teórico-práctica con aplicación á las cuestiones más usuales de aritmética, algebra, geometría, trigonometría y numerosos ejercicios y problemas, etc. Madrid, Asilo, 1906. 137 pp. P. 3.00

Grévy (A.). Géométrie dans l'espace, à l'usage des élèves des classes de première C et D (programme du 27 juillet 1905). Paris, Vuibert, 1906. 8vo. 214 pp. Fr. 2.00

Hartenstein. Resultate und Auflösungen zu "Bardey-Hartenstein, Arithmetische Aufgaben nebst Lehrbuch der Arithmetik, 6te Auflage und folgende." Leipzig, Teubner, 1906. 8vo. 80 pp. M. 1.80

Jones (A. C.) and Blomfield (C. H.). Test papers in elementary mathematics. London, Arnold, 1906. 8vo. 256 pp. 2s. 6d.

Kambly und Roeder. Stereometrie und sphärische Trigonometrie. Nach den preussischen Lehrplänen von 1901 umgearbeitet. Ausgabe der Stereometrie und der sphärischen Trigonometrie von Kambly. Lehraufgabe der Prima. Mit Uebungsaufgaben und einem Anhang: Der Koordinatenbegriff und einige Grundeigenschaften der Kegelschnitte. 4te, durchgesehene Auflage. (31te der Kamblyschen Stereometrie). Breslau, Hirt, 1906. 8vo. 224 pp. M. 2.30

Krebs (A.). Allgemeine Arithmetik in elementarer Darstellung, für den Mittelschul- und Selbstunterricht in entwickelnder Lehrform bearbeitet. 2tes Heft. Bern, Baumgart, 1906. 8vo. 3 + pp. 81–147. M. 1.30

London University matriculation mathematics papers. From Jan. 1896 to Jan. 1906. (*University Tutorial Series.*) London, Clive, 1906. 8vo. 120 pp. 1s. 6d.

Marks (C. I.). Mathematical questions and solutions. From the "Educational Times." New series. Vol. 9. London, Hodgson, 1906. 8vo. Cloth. 6s. 6d.

Maurer (H.). Methodisch geordnete Sammlung geometrischer Aufgaben in bildlicher Darstellung. 3360 Aufgaben in 4 Bänden. Zum Selbststudium und zum Unterricht an höheren Lehranstalten. 1ter Band (Aufgaben 1–840). Zürich, Speidel, 1906. 8vo. M. 2.50

M. (F. G.). Compléments de trigonométrie et méthodes pour la résolution des problèmes. Paris, Poussielgue, 1906. 8vo. 16 + 800 pp.

Müller (H.) und Plath (J.). Ergebnisse zu der Aufgabensammlung. Leipzig, Teubner, 1906. 8vo. 60 pp. M. 1.40

Plath (J). See Müller (H.).

Reinhard (A. I.). High school algebra. With answers. New York, American Book Co., 1906. 12mo. 422 pp. Cloth. $1.00

Robbins (E. R.) and Somerville (F. H.). Exercises in algebra. New York, American Book Co., 1906. 12mo. 173 pp. Half leather. $0.50

——. Answers to exercises in algebra. New York, American Book Co., 1906. $0.10

ROEDER. See KAMBLY.

SANDERS (A.). Plane and solid geometry. New York, American Book Co., 1906. $1.25

SANNIA (A.) e D'OVIDIO (E.). Elementi di geometria. Volume 1, ad uso dei ginnasii. 12a edizione. Napoli, 1906. 8vo. 16 + 200 pp. L. 2.00

SCOTTI (G.). Elementi di geometria intuitiva ad uso del ginnasio inferiore e dei corsi complementari. 8a edizione. Torino, Salesiana, 1906. 16mo. 139 pp. L. 1.00

SOMERVILLE (F. H.). See ROBBINS (E. R.).

WENTWORTH (G. A.). Elementary algebra. Boston, Ginn, 1906. 12mo. 10 + 421 pp. Half leather. $1.12

WORKMAN (W. P.) and CRACKNELL (A. G.). Geometry. Preliminary certificate edition. For course A. London, Clive, 1906. 8vo. 380 pp. Cloth. 3s. 6d.

III. APPLIED MATHEMATICS.

CALDARERA (F.). Corso di meccanica razionale. Volume III: Equilibrio e moto dei sistemi continui; idrostatica; idrodinamica. Palermo, 1906. 8vo. 268 pp. L. 11.00

CONSTAN (P.). Tables graphiques d'azimut. Paris, Gauthier-Villars, 1906. 4to. 50 pp. Fr. 3.00

DALBY (W. E.). Balancing of engines. 2d edition. London, Arnold, 1906. 8vo. 316 pp. Cloth. 10s. 6d.

FERRIS (C. E.). Elements of descriptive geometry. New York, American Book Co., 1906. 8vo. 135 pp. Cloth. $1.25

FLEMING (J. A.). Principles of electric wave telegraphy. London, Longmans, 1906. 8vo. 692 pp. Cloth. 24s.

GALILEI (G.). Le opere. Edizione nazionale sotto gli auspicî di Sua Maestà il Re d'Italia. Volume XVII. Firenze, Barbèra, 1906. 4to. 438 pp.

HOUSUM (C.). See TRINKS (W.).

KÜTTNER (W.). Das Risiko der Lebensversicherungsanstalten und Unterstützungskassen. Berlin, 1906. 8vo. 95 pp. M. 4.00

MARTIN (L. A.). Textbook of mechanics. Volume 1: Statics. New York, Wiley, 1906. 12mo. 12 + 142 pp. Cloth. $1.25

MORGAN (J. L. R.). Physical chemistry for electrical engineers. New York, Wiley, 1906. 12mo. 8 + 230 pp. Cloth. $1.50

MOULTON (F. R.). An introduction to astronomy. New York, Macmillan, 1906. 8vo. 18 + 557 pp. Cloth. $1.25

ORLICH (E.). Aufnahme und Analyse der Wechselstromkurven. Braunschweig, 1906. 8vo. 8 + 117 pp. M. 3.50

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